Direction Finding Using Sparse Linear Arrays with Missing Data

Mianzhi Wang, Zhen Zhang, and Arye Nehorai

Preston M. Green Department of Electrical & Systems Engineering
Washington University in St. Louis

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Outline

- Problem formulation
- Estimation algorithms
- Cramér-Rao bound
- Numerical examples
- Summary and future work

Notations

$$
\begin{aligned}
\boldsymbol{A}^H &= \text{Hermitian transpose of } \boldsymbol{A} \\
\boldsymbol{A}^* &= \text{Conjugate of } \boldsymbol{A} \\
\otimes &= \text{Kronecker Product} \\
\odot &= \text{Khatri-Rao Product} \\
\mathrm{vec}(\boldsymbol{A}) &= \text{Vectorization of } \boldsymbol{A} \\
\Re(\boldsymbol{A}) &= \text{Real part of} \boldsymbol{A} \\
\Im(\boldsymbol{A}) &= \text{Imaginary part of} \boldsymbol{A}
\end{aligned}
$$

Preliminaries

We consider a M-sensor sparse linear array whose sensors are located on a uniform grid, and denote the sensor locations by the integer set $\bar{\mathcal{D}} = \{\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_M\}$.

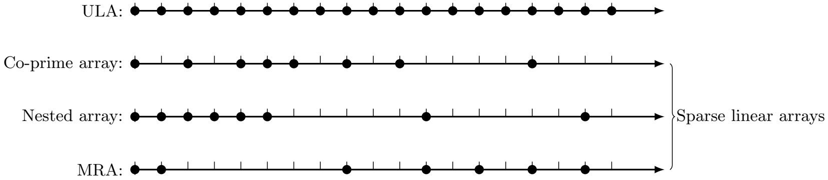

Figure 1: Examples of sparse linear arrays.

Preliminaries (cont.)

- We consider the classical far-field narrow-band measurement model:

$$\boldsymbol{y}(t) = \boldsymbol{S}\boldsymbol{A}_{\mathrm{U}}(\boldsymbol{\theta})\boldsymbol{x}(t) + \boldsymbol{n}(t),\ t = 1, 2, \ldots, N, \quad (1)$$

  where $\boldsymbol{A}_{\mathrm{U}}(\boldsymbol{\theta}) = [\boldsymbol{a}_{\mathrm{U}}(\theta_1), \boldsymbol{a}_{\mathrm{U}}(\theta_2), \ldots, \boldsymbol{a}_{\mathrm{U}}(\theta_K)]$ is the steering matrix of a M_0-sensor ULA, $M_0 = \bar{d}_M - \bar{d}_1 + 1$, $\boldsymbol{S}$ is a $M \times M_0$ selection matrix that converts a ULA manifold to a sparse linear array manifold, $x(t)$ is the source signal, and $n(t)$ is the additive noise.

- Assumptions:

  1. The source signals are temporally and spatially uncorrelated.
  2. The noise is temporally and spatially uncorrelated Gaussian that is also uncorrelated from the source signals.
  3. The K DOAs are distinct.

Preliminaries (cont.)

- The sample covariance matrix is given by:

$$\boldsymbol{R} = \mathbb{E}[\boldsymbol{y}(t)\boldsymbol{y}^H(t)] = \boldsymbol{S}\boldsymbol{R}_{\mathrm{U}}\boldsymbol{S}^T, \tag{2}$$

where $\boldsymbol{R}_{\mathrm{U}} = \boldsymbol{A}_{\mathrm{U}}\boldsymbol{P}\boldsymbol{A}_{\mathrm{U}}^H + \sigma_{\mathrm{n}}^2\boldsymbol{I}$, $\boldsymbol{P} = \mathrm{diag}(p_1, p_2, \ldots, p_K)$, and p_k is the power of k-th source.

- We can vectorize $\boldsymbol{R}$ and obtain

$$\boldsymbol{r} = (\boldsymbol{S} \otimes \boldsymbol{S})(\boldsymbol{A}_{\mathrm{U}}^* \odot \boldsymbol{A}_{\mathrm{U}})\boldsymbol{p} + \sigma_{\mathrm{n}}^2\boldsymbol{i}, \tag{3}$$

where $\boldsymbol{r} = \mathrm{vec}(\boldsymbol{R})$, $\boldsymbol{p} = [p_1, p_2, \cdots, p_K]^T$, and $\boldsymbol{i} = \mathrm{vec}(\boldsymbol{I})$.

- Model (3) resembles a difference coaray model with deterministic sources and noise, and $(\boldsymbol{S} \otimes \boldsymbol{S})(\boldsymbol{A}_{\mathrm{U}}^* \odot \boldsymbol{A}_{\mathrm{U}})$ embeds a steering matrix of a virtual array with enhanced degrees of freedom, whose sensor locations are given by $\bar{\mathcal{D}}_{\mathrm{co}} = \{(\bar{d}_m - \bar{d}_n) | \bar{d}_m, \bar{d}_n \in \bar{\mathcal{D}}\}$ [1], [2].

Preliminaries (cont.)

Definition 1. *A sparse linear array is called complete if its difference coarray $\bar{\mathcal{D}}_{\mathrm{co}}$ consists of consecutive integers from $-M_0 + 1$ to $M_0 - 1$. Otherwise, we call the sparse linear array incomplete.*

Example 1. Nested arrays [1] and minimum redundancy linear arrays [3] are complete sparse linear arrays. Co-prime arrays [4] are generally incomplete sparse linear arrays.

Implications:

- For complete arrays, we can reconstruct $\boldsymbol{R}_{\mathrm{U}}$ from the estimate of $\boldsymbol{R}$. We can identify more sources than the number of sensors.
- For incomplete arrays, we can only reconstruct a submatrix of $\boldsymbol{R}_{\mathrm{U}}$ from the estimate of $\boldsymbol{R}$. We can still resolve more sources than the number of sensors if the dimension of the submatrix is large enough.

For brevity, we restrict our following discussion to complete arrays, which can be easily extended to handle incomplete arrays.

Missing Data Problem Formulation

- We consider L sampling periods. Without loss of generality, we assume that sensor failure only occurs after the first sampling period. If a sensor fails, it will not recover in the following periods.
- We denote the valid snapshots taken during the l-th period by

$$\boldsymbol{y}_l(t) = \boldsymbol{T}_l[\boldsymbol{S}\boldsymbol{A}_{\mathrm{U}}(\boldsymbol{\theta})\boldsymbol{x}(t) + \boldsymbol{n}(t)], \quad (4)$$

  for $t = N_1 + \cdots + N_{l-1} + 1, \ldots, N_1 + \cdots + N_{l-1} + N_l$, where N_l is the number of snapshots collected during the l-th period, and $\boldsymbol{T}_l$ is a selection matrix that selects the valid sensors.
- Goal: estimate the DOAs from the measurements $\boldsymbol{y}_l(t)$.
  Problem: the coarray structure is destroyed due to sensor failures.

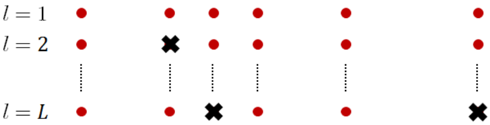

Figure 2: An example of the sensor failure pattern.

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General Idea

- The sample covariance matrix of the l-th period is given by

$$\boldsymbol{R}_l = \mathbb{E}[\boldsymbol{y}_l(t)\boldsymbol{y}_l(t)^H] = \boldsymbol{T}_l\boldsymbol{S}\boldsymbol{R}_{\mathrm{U}}\boldsymbol{S}^T\boldsymbol{T}_l^T + \sigma_{\mathrm{n}}^2\boldsymbol{I}, \tag{5}$$

which is actually formed by deleting rows and columns in $\boldsymbol{R}_{\mathrm{U}}$, as illustrated in Fig. 3.

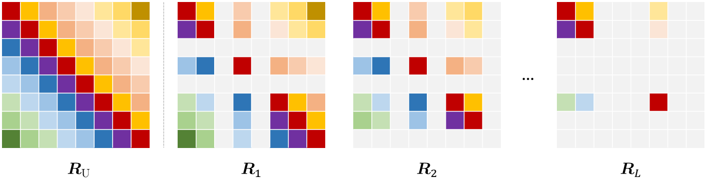

Figure 3: A illustration of the relationships between $\boldsymbol{R}_{\mathrm{U}}$ and $\boldsymbol{R}_1, \ldots, \boldsymbol{R}_L$.

- We hope to recover $\boldsymbol{R}_{\mathrm{U}}$ from the estimates $\hat{\boldsymbol{R}}_1, \ldots, \hat{\boldsymbol{R}}_L$, and estimate the DOAs based on the reconstructed $\boldsymbol{R}_{\mathrm{U}}$.

Ad-hoc Estimator

Idea: recover $\boldsymbol{R}_{\mathrm{U}}$ by averaging the elements in $\hat{\boldsymbol{R}}_l$.

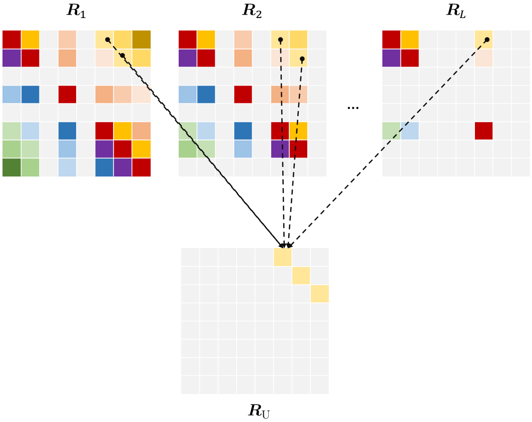

Figure 4: The idea of the ad-hoc estimator.

Ad-hoc Estimator (cont.)

- Extending the results in [5], let $\mathcal{V}_k = \{(m,n) | \bar{d}_m - \bar{d}_n = k, \bar{d}_m, \bar{d}_n \in \bar{\mathcal{D}}\}$. Let $\mathcal{A}_{m,n}$ denote the set of snapshot indices when both the m-th and the n-th sensor are working.
- Define

$$u_k = \frac{\sum_{(m,n)\in\mathcal{V}_k} \sum_{t\in\mathcal{A}_{m,n}} y_m(t) y_n^*(t)}{\sum_{(m,n)\in\mathcal{V}_k} |\mathcal{A}_{m,n}|}, \tag{6}$$

  where $\boldsymbol{y}(t) = [y_1(t), \cdots, y_M(t)]$ is the full measurement vector before discarding invalid data, and $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$.
- We can obtain u_k for $k = -M_0+1, -M_0+2, \ldots, M_0-1$, and the ad-hoc estimate of $\boldsymbol{R}_\mathrm{U}$ is given by

$$\hat{\boldsymbol{R}}_\mathrm{U}^{(\mathrm{ad-hoc})} = \begin{bmatrix} u_0 & u_{-1} & \cdots & u_{-M_0+1} \\ u_1 & u_0 & \cdots & u_{-M_0+2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{M_0} & u_{M_0-1} & \cdots & u_0 \end{bmatrix}. \tag{7}$$

Maximum-Likelihood Based Estimator

- Neglecting constant terms, the negative log-likelihood function is given by

$$L(\boldsymbol{R}_1, \ldots, \boldsymbol{R}_L) = \sum_{l=1}^{L} N_l [\log |\boldsymbol{R}_l| + \operatorname{tr}(\boldsymbol{R}_l^{-1} \hat{\boldsymbol{R}}_l)], \tag{8}$$

- We adopt the Toeplitz parameterization of $\boldsymbol{R}_{\mathrm{U}}$ [6]:

$$\boldsymbol{R}_{\mathrm{U}} = \sum_{i=1}^{2M_0-1} c_i \boldsymbol{Q}_{M_0}^{(i)}, \tag{9}$$

where the basis matrices $\boldsymbol{Q}_{M_0}^{(i)}$ are given by

$$\boldsymbol{Q}_{M_0}^{(i)} = \begin{cases} \boldsymbol{I}_{M_0}, & i = 1, \\ \boldsymbol{I}_{M_0}^{(i-1)} + (\boldsymbol{I}_{M_0}^{(i-1)})^T, & 2 \leq i \leq M_0, \\ -j\boldsymbol{I}_{M_0}^{(i-M_0)} + j(\boldsymbol{I}_{M_0}^{(i-M_0)})^T, & M_0 + 1 \leq i \leq 2M_0 - 1. \end{cases} \tag{10}$$

Remark 1. Positive semidefinite Toeplitz matrices can be related to DOAs via Vandermonde decomposition. However, this relationship is not one-to-one. Hence we are relaxing the parameter space.

Maximum-Likelihood Based Estimator (cont.)

- The partial derivatives w.r.t. the Toeplitz parameterization are give by

$$\frac{\partial L(\boldsymbol{c})}{\partial c_i} = \sum_{l=1}^{L} N_l \operatorname{tr}\left\{ \boldsymbol{T}_l \boldsymbol{S} \boldsymbol{Q}_{M_0}^{(i)} \boldsymbol{S}^T \boldsymbol{T}_l^T \boldsymbol{R}_l^{-1}(\boldsymbol{c}) \left[\boldsymbol{R}_l(\boldsymbol{c}) - \hat{\boldsymbol{R}}_l \right] \boldsymbol{R}_l^{-1}(\boldsymbol{c}) \right\} \tag{11}$$

- Let $\boldsymbol{Q}_{M_0} = [\boldsymbol{q}_{M_0}^{(1)}, \boldsymbol{q}_{M_0}^{(2)}, \cdots, \boldsymbol{q}_{M_0}^{(2M_0-1)}]$, where $\boldsymbol{q}_{M_0}^{(i)} = \operatorname{vec} \boldsymbol{Q}_{M_0}^{(i)}$. From the first-order optimality condition of (8), we can obtain the following approximate solution by replace some $\boldsymbol{W}_l = \boldsymbol{R}_l^T \otimes \boldsymbol{R}_l$ with their estimates:

$$\hat{\boldsymbol{c}}_{\mathrm{WLS}} = \left[\sum_{l=1}^{L} N_l \hat{\boldsymbol{G}}_l \right]^{-1} \left[\sum_{l=1}^{L} N_l \hat{\boldsymbol{h}}_l \right], \tag{12}$$

where $\boldsymbol{G}_l = \boldsymbol{Q}_{M_0}^T \boldsymbol{\Phi}_l^T \hat{\boldsymbol{W}}_l^{-1} \boldsymbol{\Phi}_l \boldsymbol{Q}_{M_0}$, $\boldsymbol{h}_l = \boldsymbol{Q}_{M_0}^T \boldsymbol{\Phi}_l^T \hat{\boldsymbol{W}}_l^{-1} \hat{\boldsymbol{r}}_l$, and $\hat{\boldsymbol{W}}_l = \hat{\boldsymbol{R}}_l^T \otimes \hat{\boldsymbol{R}}_l$. This solution is also the solution to the weighted least-squares problem:

$$\min_{\boldsymbol{c}} \sum_{l=1}^{L} N_l \| \boldsymbol{\Phi}_l \boldsymbol{Q}_{M_0} \boldsymbol{c} - \hat{\boldsymbol{r}}_l \|_{\hat{\boldsymbol{W}}_l^{-1}}^2 \tag{13}$$

Without replacing any $\boldsymbol{W}_l$ with $\hat{\boldsymbol{W}}_l$, the first-order optimality condition of (8) also leads to the following fixed-point iteration procedure:

$$\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k)} = \left[\sum_{l=1}^{L} N_l \boldsymbol{G}_l\left(\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k-1)}\right)\right]^{-1}\left[\sum_{l=1}^{L} N_l \boldsymbol{h}_l\left(\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k-1)}\right)\right], \tag{14}$$

where

$$\boldsymbol{G}_l\left(\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k-1)}\right) = \boldsymbol{Q}_{M_0}^T \boldsymbol{\Phi}_l^T \boldsymbol{W}_l^{-1}\left(\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k-1)}\right)\boldsymbol{\Phi}_l \boldsymbol{Q}_{M_0}, \tag{15a}$$

$$\boldsymbol{h}_l\left(\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k-1)}\right) = \boldsymbol{Q}_{M_0}^T \boldsymbol{\Phi}_l^T \boldsymbol{W}_l^{-1}\left(\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k-1)}\right)\hat{\boldsymbol{r}}_l, \tag{15b}$$

$$\boldsymbol{W}_l\left(\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k-1)}\right) = \hat{\boldsymbol{W}}_l = \hat{\boldsymbol{R}}_l^T\left(\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k-1)}\right) \otimes \hat{\boldsymbol{R}}_l\left(\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k-1)}\right). \tag{15c}$$

Remark 2. The fixed-point iteration (14) can be initialized with $\hat{\boldsymbol{c}}_{\mathrm{WLS}}$ and produces good estimates within several iterations in our simulations.

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Cramér-Rao Bound

- We derive the CRB for both the Toeplitz parameterization and the DOA parameterization based on classical results in [7], [8].
- For complete arrays, the FIM for the Toeplitz parameterization is given by

$$\mathrm{FIM}_{\boldsymbol{c}} = \sum_{l=1}^{L} N_l \boldsymbol{Q}_{M_0}^H \boldsymbol{\Phi}_l^H (\boldsymbol{R}_l^T \otimes \boldsymbol{R}_l)^{-1} \boldsymbol{\Phi}_l \boldsymbol{Q}_{M_0}. \quad (16)$$

- For complete arrays, the FIM of the parameters $\boldsymbol{\eta} = [\boldsymbol{\theta}, \boldsymbol{p}, \sigma_{\mathrm{n}}^2]^T$ is given by

$$\mathrm{FIM}_{\boldsymbol{\eta}} = \sum_{l=1}^{L} N_l \boldsymbol{D}^H \boldsymbol{\Phi}_l^H (\boldsymbol{R}_l^T \otimes \boldsymbol{R}_l)^{-1} \boldsymbol{\Phi}_l \boldsymbol{D}, \quad (17)$$

where $\boldsymbol{D} = [\dot{\boldsymbol{A}}_{\mathrm{d}} \boldsymbol{P} \; \boldsymbol{A}_{\mathrm{d}} \; \boldsymbol{i}]$, and $\dot{\boldsymbol{A}}_{\mathrm{d}} = \dot{\boldsymbol{A}}_{\mathrm{U}}^* \odot \boldsymbol{A}_{\mathrm{U}} + \boldsymbol{A}_{\mathrm{U}}^* \odot \dot{\boldsymbol{A}}_{\mathrm{U}}$,
$\dot{\boldsymbol{A}}_{\mathrm{U}} = [\partial \boldsymbol{a}_{\mathrm{U}}(\theta_1)/\partial\theta_1, \cdots, \partial \boldsymbol{a}_{\mathrm{U}}(\theta_K)/\partial\theta_K]$, $\boldsymbol{A}_{\mathrm{d}} = \boldsymbol{A}_{\mathrm{U}}^* \odot \boldsymbol{A}_{\mathrm{U}}$, and $\boldsymbol{i} = \mathrm{vec}(\boldsymbol{I}_{M_0})$.

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Experiment Setup

- We consider the following two arrays:
  - Nested array: $[0, 1, 2, 3, 7, 11, 15, 19]d_0$;
  - Coprime array: $[0, 3, 5, 6, 9, 10, 12, 15, 20, 25]d_0$.
- We consider 12 sources uniformly distributed between $-\pi/3$ and $\pi/3$, which is more than the number of sensors of either array.
- We set L to be 3. When $L = 2$ the last sensor of each array fails, and when $L = 3$, the last two sensors of each array fail.
- We set $N_1 = 50\mu$, $N_2 = 100\mu$, and $N_3 = 150\mu$, where μ is a tunable parameter.
- When making comparisons under different numbers of snapshots, we fixed $\mathrm{SNR} = 0\mathrm{dB}$ and varied μ from 1 to 20. When making comparisons under different SNRs, we fixed $\mu = 1$ and varied SNR from -20dB to 20dB.
- We compare four method of estimating $\boldsymbol{R}_{\mathrm{U}}$ for DOA estimation: using the complete data only, using the ad-hoc estimator (7), using (12), and using (14).

Numerical Examples for the Nested Array

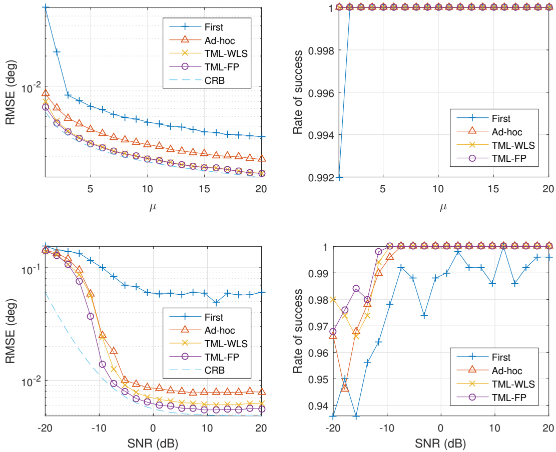

Figure 5: Performance of different algorithms for the nested array configuration.

Numerical Examples for the Co-Prime Array

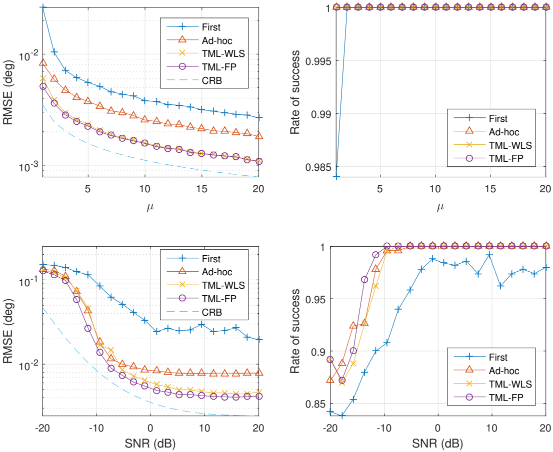

Figure 6: Performance of different algorithms for the co-prime array configuration.

Observations

- Performance gain is significant by utilizing the incomplete measurements in addition to the complete measurements.
- The weighted least squares estimator (12) (TML-WLS) and the fixed-point iteration based estimator (14) (TML-FP) achieves lower estimation errors than the ad-hoc estimator.
- In the nested array case, the RMSEs of TML-WLS and TML-FP is very close to the CRB. In the co-prime array case, there is always a gap between the RMSEs and the CRB.

Summary and Future Work

Summary and Future Work

Summary:

- We proposed ML-based methods to reconstruct a sample covariance matrix with enhanced degrees of freedom using the Toeplitz parameterization in the missing data case, which enables us to resolve more sources than the number of sensors using sparse linear arrays.
- We derived the corresponding CRBs for sparse linear arrays.

Future Work:

- Detection of malfunctioning sensors
- Performance analysis in the presence of sensor failures

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Questions?

Other Formulations

- Measurement interpolation.
  - Computationally expensive when the number of snapshots is larger.
- Low-rank Toeplitz matrix completion via nuclear norm minimization.
  - Requires solving a SDP problem.