Performance Analysis of Coarray-Based MUSIC and the Cramér-Rao Bound

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Outline

- Measurement model and coarray-based MUSIC
- Mean-square error of coarray-based MUSIC
- Cramér-Rao bound
- Conclusions and future work

Notations

$$
\begin{aligned}
\boldsymbol{A}^H &= \text{Hermitian transpose of } \boldsymbol{A} \\
\boldsymbol{A}^* &= \text{Conjugate of } \boldsymbol{A} \\
\boldsymbol{A}^\dagger &= (\boldsymbol{A}^H\boldsymbol{A})^{-1}\boldsymbol{A}^H, \text{ pseudo inverse of } \boldsymbol{A} \\
\boldsymbol{\Pi}_{\boldsymbol{A}} &= \boldsymbol{A}\boldsymbol{A}^\dagger, \text{ projection matrix onto the range space of } \boldsymbol{A} \\
\boldsymbol{\Pi}_{\boldsymbol{A}}^\perp &= \boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^\dagger, \text{ projection matrix onto the null space of } \boldsymbol{A} \\
\otimes &= \text{Kronecker Product} \\
\odot &= \text{Khatri-Rao Product} \\
\mathrm{vec}(\boldsymbol{A}) &= \text{Vectorization of } \boldsymbol{A} \\
\Re(\boldsymbol{A}) &= \text{Real part of } \boldsymbol{A} \\
\Im(\boldsymbol{A}) &= \text{Imaginary part of} \boldsymbol{A}
\end{aligned}
$$

Measurement Model

- We consider a far-field narrow-band measurement model of sparse linear arrays:

$$\boldsymbol{y}(t) = \boldsymbol{A}(\theta)\boldsymbol{x}(t) + \boldsymbol{n}(t), \tag{1}$$

where $\boldsymbol{A} = [\boldsymbol{a}(\theta_1)\, \boldsymbol{a}(\theta_2)\, \cdots\, \boldsymbol{a}(\theta_K)]$, with the i-th element of $\boldsymbol{a}(\theta_k)$ being $e^{j\bar{d}_i\phi_k}$, $\bar{d}_i = d_i/d_0$, $\phi_k = (2\pi d_0 \sin\theta_k)/\lambda$, and λ denotes the wavelength.

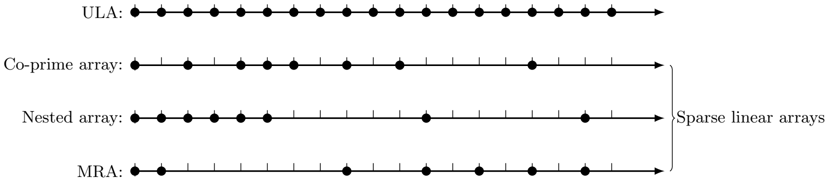

Figure 1: Examples of sparse linear arrays.

Measurement Model (cont.)

- We consider the stochastic (unconditional) model [1], where the sources signals are assumed random and unknown.
- Assumptions:
  1. The source signals are temporally and spatially uncorrelated.
  2. The noise is temporally and spatially uncorrelated Gaussian that is also uncorrelated from the source signals.
  3. The K DOAs are distinct.
- The sample covariance matrix is given by

$$\boldsymbol{R} = \mathbb{E}[\boldsymbol{y}\boldsymbol{y}^H] = \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^H + \sigma_{\mathrm{n}}^2\boldsymbol{I}, \tag{2}$$

where $\boldsymbol{P} = \mathrm{diag}(p_1, p_2, \ldots, p_L)$ is the source covariance matrix.

Coarray-based MUSIC

- Vectorizing $\boldsymbol{R}$ leads to

$$\boldsymbol{r} = \operatorname{vec} \boldsymbol{R} = \boldsymbol{A}_{\mathrm{d}} \boldsymbol{p} + \sigma_{\mathrm{n}}^2 \boldsymbol{i}, \tag{3}$$

where $\boldsymbol{p} = [p_1, p_2, \ldots, p_K]^T$, $\boldsymbol{i} = \operatorname{vec}(\boldsymbol{I})$, and

$$\boldsymbol{A}_{\mathrm{d}} = \boldsymbol{A}^* \odot \boldsymbol{A} = \begin{bmatrix} e^{j(\bar{d}_1 - \bar{d}_1)\phi_1} & \cdots & e^{j(\bar{d}_1 - \bar{d}_1)\phi_k} \\ \vdots & \ddots & \vdots \\ e^{j(\bar{d}_m - \bar{d}_n)\phi_1} & \cdots & e^{j(\bar{d}_m - \bar{d}_n)\phi_k} \\ \vdots & \ddots & \vdots \\ e^{j(\bar{d}_M - \bar{d}_M)\phi_1} & \cdots & e^{j(\bar{d}_M - \bar{d}_M)\phi_k} \end{bmatrix}. \tag{4}$$

- **Observation:** $\boldsymbol{A}_{\mathrm{d}}$ embeds a steering matrix of an difference coarray whose sensor locations are given by $\mathcal{D}_{\mathrm{co}} = \{d_m - d_n | 1 \leq m, n \leq M\}$.

  $\Rightarrow$ We can construct a virtual ULA model from (3).

Coarray-based MUSIC (cont.)

Example 1. An illustration of the relationship between the physical array and the difference coarray.

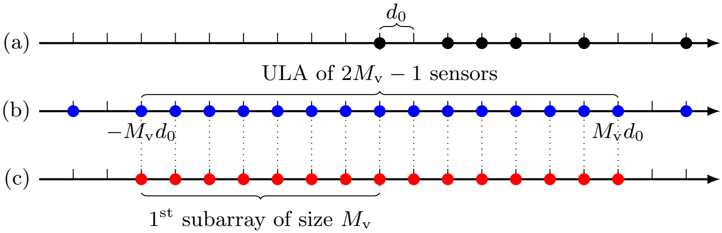

Figure 2: A co-prime array with sensors located at $[0, 2, 3, 4, 6, 9]\lambda/2$ and its coarray: (a) physical array, (b) coarray, (c) virtual ULA part of the coarray.

Coarray-based MUSIC (cont.)

Definition 1. The array weight function [2] $\omega(n) : \mathbb{Z} \mapsto \mathbb{Z}$ is defined by $\omega(l) = |\{(m,n)|\bar{d}_m - \bar{d}_n = l\}|$, where $|\mathcal{A}|$ denotes the cardinality of the set $\mathcal{A}$.

Definition 2. Let $2M_\mathrm{v} - 1$ denote the size of the central virtual ULA. We introduce the transform matrix [3] $\boldsymbol{F}$ as a real matrix of size $(2M_\mathrm{v} - 1) \times M^2$, whose elements are defined by

$$F_{m,p+(q-1)M} = \begin{cases} \frac{1}{\omega(m-M_\mathrm{v})} & , \bar{d}_p - \bar{d}_q = m - M_\mathrm{v}, \\ 0 & , \text{otherwise}, \end{cases} \quad (5)$$

for $m = 1, 2, \ldots, M_\mathrm{v}, p = 1, 2, \ldots, M, q = 1, 2, \ldots, M$.

$\Rightarrow$ We can express the measurement vector of the virtual ULA model by

$$\boldsymbol{z} = \boldsymbol{F}\boldsymbol{r} = \boldsymbol{A}_\mathrm{c}\boldsymbol{p} + \sigma_\mathrm{n}^2\boldsymbol{F}\boldsymbol{i}. \quad (6)$$

Coarray-based MUSIC (cont.)

- To construct the augmented sample covariance matrix, the virtual ULA is divided into M_v overlapping subarrays of size M_v [2], [4].
- We denote the output of the i-th subarray by $\boldsymbol{z}_i = \boldsymbol{\Gamma}_i \boldsymbol{z}$ for $i = 1, 2, \ldots, M_\mathrm{v}$, where $\boldsymbol{\Gamma}_i = [\mathbf{0}_{M_\mathrm{v} \times (i-1)} \ \boldsymbol{I}_{M_\mathrm{v} \times M_\mathrm{v}} \ \mathbf{0}_{M_\mathrm{v} \times (M_\mathrm{v}-i)}]$.

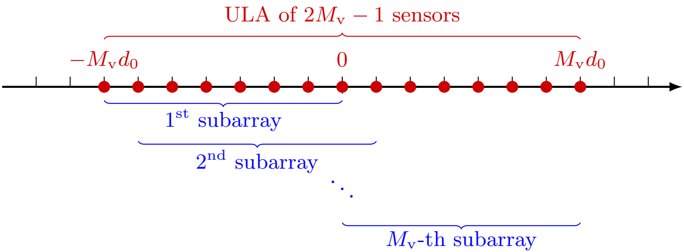

Figure 3: M_v overlapping subarrays.

Coarray-based MUSIC (cont.)

- We can then construct an augmented covariance matrix $\boldsymbol{R}_{\mathrm{v}}$ from $\boldsymbol{z}_i$ to provide enhanced degrees of freedom and apply MUSIC to $\boldsymbol{R}_{\mathrm{v}}$.
- Two commonly used methods:
  - MUSIC with directly augmented covariance matrix (DA-MUSIC) [4]:

$$\boldsymbol{R}_{\mathrm{v}1} = [\boldsymbol{z}_{M_{\mathrm{v}}}\ \boldsymbol{z}_{M_{\mathrm{v}}-1}\ \cdots\ \boldsymbol{z}_1]. \tag{7}$$

  - MUSIC with spatially smoothed covariance matrix (SS-MUSIC) [2]:

$$\boldsymbol{R}_{\mathrm{v}2} = \frac{1}{M_{\mathrm{v}}} \sum_{i=1}^{M_{\mathrm{v}}} \boldsymbol{z}_i \boldsymbol{z}_i^H. \tag{8}$$

- $\boldsymbol{R}_{\mathrm{v}1}$ and $\boldsymbol{R}_{\mathrm{v}2}$ are related via the following equality [2]:

$$\boldsymbol{R}_{\mathrm{v}2} = \frac{1}{M_{\mathrm{v}}} \boldsymbol{R}_{\mathrm{v}1}^2 = \frac{1}{M_{\mathrm{v}}} (\boldsymbol{A}_{\mathrm{v}} \boldsymbol{P} \boldsymbol{A}_{\mathrm{v}}^H + \sigma_{\mathrm{n}}^2 \boldsymbol{I})^2, \tag{9}$$

where $\boldsymbol{A}_{\mathrm{v}}$ corresponds to the steering matrix of a ULA whose sensors are located at $[0, 1, \ldots, M_{\mathrm{v}} - 1]d_0$.

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Mean-Square Error of Coarray-Based MUSIC

We derive the closed-form MSE expressions for DA-MUSIC and SS-MUSIC:

Theorem 1. *Let $\hat{\theta}_k^{(\mathrm{DA})}$ and $\hat{\theta}_k^{(\mathrm{SS})}$ denote the estimated values of θ_k using DA-MUSIC and SS-MUSIC, respectively. Let $\Delta \boldsymbol{r} = \mathrm{vec}(\hat{\boldsymbol{R}} - \boldsymbol{R})$. Then [3]*

$$\hat{\theta}_k^{(\mathrm{DA})} - \theta_k \doteq \hat{\theta}_k^{(\mathrm{SS})} - \theta_k \doteq -\frac{\lambda}{2\pi d_0 p_k \cos\theta_k} \frac{\Im(\boldsymbol{\xi}^T \Delta \boldsymbol{r})}{\boldsymbol{\beta}_k^H \boldsymbol{\beta}_k}, \tag{10}$$

where $\doteq$ denotes asymptotic equality (first-order) and

$$\begin{aligned} \boldsymbol{\xi}_k &= \boldsymbol{F}^T \boldsymbol{\Gamma}^T (\boldsymbol{\beta}_k \otimes \boldsymbol{\alpha}_k), & \boldsymbol{\Gamma} &= [\boldsymbol{\Gamma}_{M_\mathrm{v}}^T \, \boldsymbol{\Gamma}_{M_\mathrm{v}-1}^T \cdots \boldsymbol{\Gamma}_1^T]^T, \\ \boldsymbol{\alpha}_k^T &= -\boldsymbol{e}_k^T \boldsymbol{A}_\mathrm{v}^\dagger, & \boldsymbol{D} &= \mathrm{diag}(0, 1, \ldots, M_\mathrm{v}), \\ \boldsymbol{\beta}_k &= \boldsymbol{\Pi}_{\boldsymbol{A}_\mathrm{v}}^\perp \boldsymbol{D} \boldsymbol{a}_\mathrm{v}(\theta_k). \end{aligned}$$

Theorem 2. *The asymptotic MSE expressions of DA-MUSIC and SS-MUSIC have the same form. Denote the asymptotic MSE of the k-th DOA by $\epsilon(\theta_k)$. We have [3]:*

$$\epsilon(\theta_k) = \frac{\lambda^2}{4\pi^2 N d_0^2 p_k^2 \cos^2\theta_k} \frac{\boldsymbol{\xi}_k^H (\boldsymbol{R} \otimes \boldsymbol{R}^T) \boldsymbol{\xi}_k}{\|\boldsymbol{\beta}_k\|_2^4}, \quad \forall k \in \{1, 2, \ldots, K\}. \tag{11}$$

Mean-Square Error of Coarray-Based MUSIC (cont.)

Theorem 1 and Theorem 2 have the following implications:

- DA-MUSIC and SS-MUSIC have the same asymptotic MSE, and they are both asymptotically unbiased.
- $\epsilon(\theta_k)$ depends on both the physical array geometry and the coarray geometry (as illustrated in Fig. 4).

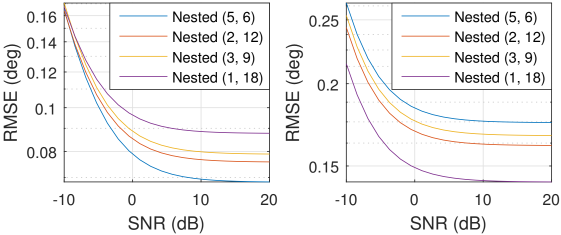


Figure 4: RMSE vs. SNR for four different nested array configurations. The four arrays share the same virtual ULA. Left: $K = 8$. Right: $K = 20$.

Mean-Square Error of Coarray-Based MUSIC (cont.)

Corollary 1. *Assume all sources have the same power* p*. Let* $\mathrm{SNR} = p/\sigma_{\mathrm{n}}^2$ *denote the common SNR. Then* $\epsilon(\theta_k)$ *decreases monotonically as* SNR *increases, and*

$$\lim_{\mathrm{SNR}\to\infty} \epsilon(\theta_k) = \frac{\lambda^2}{4\pi^2 N d_0^2 p_k^2 \cos^2\theta_k} \frac{\|\boldsymbol{\xi}_k^H(\boldsymbol{A}\otimes\boldsymbol{A}^*)\|_2^2}{\|\boldsymbol{\beta}_k\|_2^4}. \quad (12)$$

Specifically,

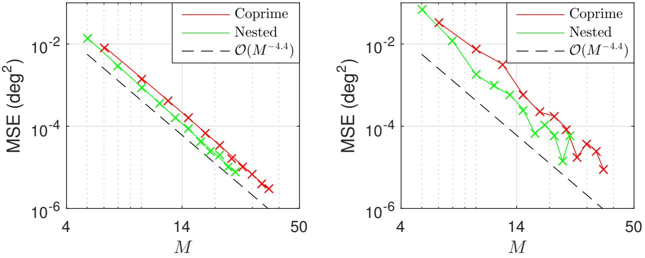
1. *when* $K = 1$*, the above expression is exactly zero;*
2. *when* $K \geq M$ *the above expression is strictly greater than zero.*

Implication:

Corollary 1 analytically explains the "saturation" behavior of SS-MUSIC in high SNR regions observed in previous studies.

Mean-Square Error of Coarray-Based MUSIC (cont.)

MSE vs. number of sensors:



(a) $K = 1$ (b) $K = 3$

Figure 5: MSE vs. number of sensors. $\mathrm{SNR} = 0\mathrm{dB}$, and $N = 1000$. The solid lines denote analytical results, while crosses denote numerical results. A dashed black trend line is included for comparison. The co-prime arrays were generated by the co-prime pairs $(m, m + 1)$, and the nested arrays were generated by the parameter pairs $(m + 1, m)$, where we varied m from 2 to 12.

Observation: the MSE of coarray-based MUSIC decreases faster than $\mathcal{O}(M^{-3})$, the asymptotic MSE of classical MUSIC for ULAs when $M \to \infty$.

Mean-Square Error of Coarray-Based MUSIC (cont.)

Resolution analysis:

The analytical resolution limit is determined by

$$\sqrt{\epsilon(\theta - \Delta\theta/2)} + \sqrt{\epsilon(\theta - \Delta\theta/2)} \geq \Delta\theta \tag{13}$$

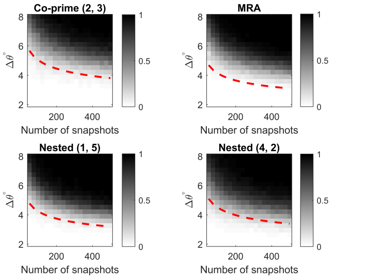


Figure 6: Resolution probability of different arrays for different N with SNR fixed to 0dB, obtained from 500 trials. The red dashed line is the analytical resolution limit.

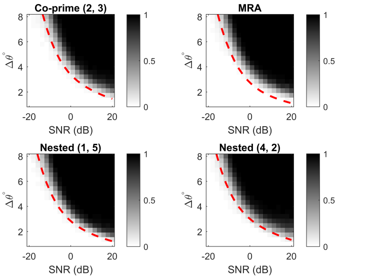


Figure 7: Resolution probability of different arrays for different SNRs with $N = 1000$, obtained from 500 trials. The red dashed line is the analytical resolution limit.

Observation: our analytical expression predict the resolution limit well.

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Cramér-Rao Bound

- The CRB of the DOAs for general sparse linear arrays with under the assumption of uncorrelated sources is given by [3], [5], [6]:

$$\mathrm{CRB}_{\boldsymbol{\theta}} = \frac{1}{N}(\boldsymbol{M}_{\boldsymbol{\theta}}^H \boldsymbol{\Pi}_{\boldsymbol{M}_{\boldsymbol{s}}}^{\perp} \boldsymbol{M}_{\boldsymbol{\theta}})^{-1}, \quad (14)$$

where

$$\boldsymbol{M}_{\boldsymbol{\theta}} = (\boldsymbol{R}^T \otimes \boldsymbol{R})^{-1/2} \dot{\boldsymbol{A}}_{\mathrm{d}} \boldsymbol{P}, \quad (15a)$$

$$\boldsymbol{M}_{\boldsymbol{s}} = (\boldsymbol{R}^T \otimes \boldsymbol{R})^{-1/2} \left[\boldsymbol{A}_{\mathrm{d}}\, \boldsymbol{i}\right], \quad (15b)$$

$$\dot{\boldsymbol{A}}_{\mathrm{d}} = \dot{\boldsymbol{A}}^* \odot \boldsymbol{A} + \boldsymbol{A}^* \odot \dot{\boldsymbol{A}}, \quad (15c)$$

$$\dot{\boldsymbol{A}} = [\partial \boldsymbol{a}(\theta_1)/\partial\theta_1, \cdots, \partial \boldsymbol{a}(\theta_K)/\partial\theta_K], \quad (15d)$$

and $\boldsymbol{A}_{\mathrm{d}}$, $\boldsymbol{i}$ follow the same definitions as in (3).

- The CRB can be valid even if the number of sources exceeds the number of sensors. This is because the invertibility of the FIM depends on the coarray structure, which appears in $[\dot{\boldsymbol{A}}_{\mathrm{d}}\boldsymbol{P}\ \boldsymbol{A}_{\mathrm{d}}\ \boldsymbol{i}]$. The can remain full column rank of $[\dot{\boldsymbol{A}}_{\mathrm{d}}\boldsymbol{P}\ \boldsymbol{A}_{\mathrm{d}}\ \boldsymbol{i}]$ even if $K \geq M$.

Cramér-Rao Bound (cont.)

Proposition 1. *Assume all sources have the same power* p*, and* $[\dot{\boldsymbol{A}}_{\mathrm{d}} \boldsymbol{P} \boldsymbol{A}_{\mathrm{d}} \boldsymbol{i}]$ *is full column rank. Let* $\mathrm{SNR} = p/\sigma_{\mathrm{n}}^2$*. Then*

1. *If* $K < M$*, and* $\lim_{\mathrm{SNR}\to\infty} \mathrm{CRB}_{\boldsymbol{\theta}}$ *exists, it is zero under mild conditions;*
2. *If* $K \geq M$*, and* $\lim_{\mathrm{SNR}\to\infty} \mathrm{CRB}_{\boldsymbol{\theta}}$ *exists, it is positive definite when* $K \geq M$*.*

Implications:

- When $K < M$, the CRB approaches zero as $\mathrm{SNR} \to \infty$, which is similar to the ULA case.
- When $K \geq M$, the CRB exhibits a different behavior by converging to a strictly positive definite matrix. This puts an strictly positive lower bound on the MSE of all unbiased estimators.
- Recall that in Corollary 1, when $K \geq M$, $\epsilon(\theta_k)$ converges to a positive constant as $\mathrm{SNR} \to \infty$. We now know that this is not because of the choice of the algorithms, but the asymptotic error $\epsilon(\theta_k) > 0$ is inherent in the model as shown by the CRB.

Cramér-Rao Bound (cont.)

Theorem 3. *Assume that all sources share the same power. For co-prime arrays generated with co-prime pair* $(Q, Q+1)$*, or nested arrays generated with parameter pair* (Q, Q)*, if we fix* $K \ll Q$*, then as* $Q \to \infty$*, the CRB can decrease at a rate of* $\mathcal{O}(Q^{-5})$*.*

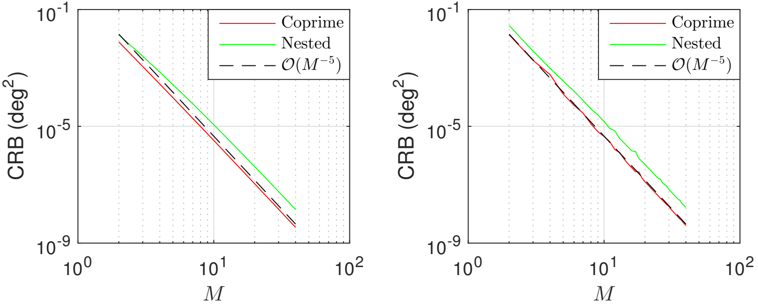
Observation:

For ULAs, the CRB decreases at a rate of $\mathcal{O}(M^{-3})$ as the number of sensors $M \to \infty$ [7]. Theorem 3 implies that co-prime and nested arrays can achieve the same performance as ULAs with fewer sensors.

This behavior can be attributed to the fact that a M-sensor co-prime array or nested array has a much larger aperture than a M-sensor ULA.

Cramér-Rao Bound (cont.)

CRB vs. number of sensors for co-prime and nested arrays:



(a) $K = 1$ (b) $K = 3$

Figure 8: CRB vs. number of sensors. $\mathrm{SNR} = 0\mathrm{dB}$, and $N = 1000$. A dashed black trend line is included for comparison.

Observation: the CRB precisely follows the trend line of $\mathcal{O}(M^{-5})$ for large M.

Conclusions and Future Work

Conclusions

- DA-MUSIC and SS-MUSIC have the same asymptotic MSE.
- When there are more sources than sensors, both the MSE of DA-MUSIC (SS-MUSIC) and the CRB are strictly non-zero as $\mathrm{SNR} \to \infty$.
- The CRB for co-prime and nested arrays with $\mathcal{O}(M)$ sensors can decrease at a rate of $\mathcal{O}(M^{-5})$, which analytically shows that such arrays can achieve similar performance to ULAs with fewer sensors.

Future work:

- Analytical resolution analysis
- Sensitivity analysis against model errors
- Optimal array geometry design

References I

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Questions?