Performance Analysis of Coarray-Based MUSIC in the Presence of Sensor Location Errors

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Abstract-Sparse linear arrays, such as co-prime and nested arrays, can resolve more uncorrelated sources than the number of sensors by applying the MUtiple SIgnal Classification (MU-SIC) algorithm to their difference coarray model. We aim at statistically analyzing the performance of the MUSIC algorithm applied to the difference coarray model, namely, the coarraybased MUSIC, in the presence of sensor location errors. We first introduce a signal model for sparse linear arrays in the presence of deterministic unknown location errors. Based on this signal model, we derive a closed-form expression of the asymptotic mean-squared error (MSE) of a commonly used coarray-based MUSIC algorithm, SS-MUSIC, in the presence of small sensor location errors. We show that the sensor location errors introduce a constant bias that depends on both the physical array geometry and the coarray geometry, which cannot be mitigated by only increasing the signal-to-noise ratio (SNR). We next give a short extension of our analysis to cases when the sensor location errors are stochastic, and investigate the Gaussian case. Finally, we derive the Cramér-Rao bound for joint estimation of directionof-arrivals (DOAs) and sensor location errors for sparse linear arrays, which can be applicable even if the number of sources exceeds the number of sensors. Numerical simulations show good agreement between empirical results and our theoretical results.

Index Terms—performance analysis, sparse arrays, co-prime arrays, nested arrays, mean-squared error, MUSIC, Cramér-Rao bound

I. INTRODUCTION

D IRECTION-OF-ARRIVAL (DOA) estimation is an important topic in array signal processing with wide applications, such as radar and sonar [1], [2]. With uniform linear arrays (ULAs), classical subspace-based algorithms, such as MUSIC, can resolve up to M-1 sources using M sensors [3]–[6]. Sparse linear arrays, such as co-prime arrays [7]–[10] and nested arrays [11]–[15], are specially designed non-uniform linear arrays. By exploiting their difference coarray model, an augmented covariance matrix with enhanced degrees of freedom can be constructed [11], [16]. MUSIC can then be applied to the augmented covariance to resolve up to $\mathcal{O}(M^2)$ uncorrelated sources using only $\mathcal{O}(M)$ sensors.

The aforementioned difference coarray model is developed under the assumption that the underlying array is accurately calibrated. However, various array imperfections, such as mutual coupling [17], gain and phase errors [18], [19], and location errors [20], exist in practice and lead to degraded

estimation performance [21], [22]. Various works have focused on analyzing the sensitivity of direction finding algorithms and the achievable bounds in the presence of array imperfections. In [20], the authors derived a hybrid Cramér-Rao bound on calibration and source localization for general two-dimensional arrays in the presence of sensor location errors. Based on the derived Cramér-Rao bound (CRB), the authors showed the condition when the CRB goes to zero as the SNR approaches infinity. In [21] and [23], the authors conducted a thorough performance analysis of subspace-based DOA estimators in the presence of model errors. In [24], the authors analyzed the resolution probability of the MUSIC algorithm, while taking into account model errors. However, the aforementioned analyses are based on the physical array model, and the number of sources is usually fewer than the number of sensors. The performance of direction finding algorithms based on the difference coarray in the presence of model errors has not been widely analyzed. Recently, in [25], the authors evaluated the performance of uniform and nonuniform samplers in the presence of model errors based on the CRB of grid-based model. These results can be applied to direction finding algorithms based on the difference coarray model. However, their analysis assumes one-dimensional perturbations along the array and that the DOAs lie on a predefined grid. In this paper, neither do we restrict our analysis to one-dimensional perturbations, nor do we assume a grid-based model.

Sensor location errors are common array imperfections. Unlike gain and phase errors, perturbed array manifolds are nonlinear with respect to sensor location errors. This nonlinearity makes it more difficult to analyze the effect of sensor location errors on direction finding algorithms. In this manuscript, we focus on analyzing the performance of coarraybased MUSIC in the presence of sensor location errors. More specifically, we consider the commonly used SS-MUSIC [11] algorithm. We first introduce a signal model of difference coarrays in the presence of deterministic sensor location errors in Section II. Based on this signal model, we derive a closed-form expression of the asymptotic (i.e., large number of snapshots) MSE of SS-MUSIC in the presence of small sensor location errors in Section III. We show that the sensor location errors result in a constant bias that depends on both the physical array geometry and the difference coarray geometry. We then provide an brief extension of our analysis to incorporate stochastic (or time-variant) sensor location errors, specifically for the Gaussian case, in Section IV. Finally, in Section V, we derive the CRB on joint estimation of the DOAs and sensor location errors for general sparse linear arrays, which can be applicable even if the number of sources exceeds the number

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of sensors. We present extensive numerical demonstrations in Section VI and draw concluding remarks in Section VII. It should be mentioned that while our analyses are focused on sensor location errors, they can be readily extended to incorporate other array imperfections.

Throughout this paper, we make use of the following notations. For a matrix A, we denote the transpose, the Hermitian transpose, and the conjugate of A by A^T , A^H , and A^* , respectively. We use A_{ij} or A(i,j) to denote the (i,j)-th element of A, and a_i to denote the *i*-th column of A. If A is full column rank, its pseudo inverse is defined as $A^{\dagger} = (A^H A)^{-1} A^H$. We define the projection matrix onto the null space of A as $\Pi_A^{\perp} = I - AA^{\dagger}$. Let $A = [a_1 a_2 \dots a_N] \in \mathbb{C}^{M \times N}$, and we define the vectorization operation as $\operatorname{vec}(A) = [a_1^T a_2^T \dots a_N^T]^T$. We use \otimes , \odot , and \circ to denote the Kronecker product, the Khatri-Rao product (i.e., the column-wise Kronecker product), and the Hadamard product, respectively. We denote by $\Re(A)$ and $\Im(A)$ the real and the imaginary parts of A. If A is a square matrix, we denote its trace by $\operatorname{tr}(A)$. We use $e_N^{(i)}$ to denote the *i*-th natural base vector in \mathbb{R}^N .

II. SIGNAL MODEL

A sparse linear array can be viewed as a thinned ULA, whose sensors are located on a uniform grid of grid size d_0 , where d_0 is usually chosen as the half-wavelength, $\lambda/2$, to avoid ambiguities. Examples of sparse linear arrays include coprime arrays [7], nested arrays [11], and minimum redundancy linear arrays (MRAs) [26]. The sensor locations of co-prime arrays and nested arrays can be determined by closed-form expressions, as presented in Definition 1. Minimum redundancy linear arrays, however, can not be constructed from closedform expressions.

Definition 1. A co-prime array generated by the co-prime pair (M, N) is given by $\{0, M, \ldots, (N-1)M\} \cup \{N, 2N, \ldots, (2M-1)N\}$ [7]. A nested array generated by the parameter pair (N_1, N_2) is given by $\{0, 1, \ldots, N_1 - 1\} \cup \{N_1, 2N_1 + 1, \ldots, N_2N_1 + N_2 - 1\}$ [11].

We consider a sparse linear array placed along the x-axis of a 2D plane. We denote the nominal sensor locations as $\mathcal{D} = \{(d_1, 0), (d_2, 0), \dots, (d_M, 0)\}$, where M is the total number of sensors and d_i are integer multiples of d_0 . Without loss of generality, we assume $d_1 = 0$ (i.e., the first sensor is placed at the origin).

We consider K co-planar far-field narrow-band sources impinging on the array from the directions $\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_K]$. The N snapshots received by the array are expressed as

$$\boldsymbol{y}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{x}(t) + \boldsymbol{n}(t), t = 1, 2, \dots, N,$$
(1)

where $A(\theta) = [a(\theta_1) \ a(\theta_2) \ \cdots \ a(\theta_K)]$ denotes the array steering matrix, x(t) denotes source signals, and n(t) denotes additive noise. Without sensor location errors, the vector $a(\theta_k)$ can be expressed as

$$\boldsymbol{a}(\theta_k) = \left[e^{j\frac{2\pi}{\lambda}d_1\sin\theta_k} \ e^{j\frac{2\pi}{\lambda}d_2\sin\theta_k} \ \cdots \ e^{j\frac{2\pi}{\lambda}d_M\sin\theta_k}\right]^T.$$
(2)

We make the following assumptions on the statistical properties of the source and noise signals:



Fig. 1. (a) A co-prime array with sensors located at $[0, 2, 3, 4, 6, 9]d_0$; (b) its difference coarray; (c) central ULA part of the difference coarray.

- A1 The source signals follow the unconditional model [27] and are spatially and temporally uncorrelated.
- A2 The source DOAs are distinct (i.e., $\theta_k \neq \theta_l \ \forall k \neq l$) and within the range $(-\pi/2, \pi/2)$.
- A3 The additive noise is spatially and temporally uncorrelated white circularly-symmetric Gaussian and uncorrelated from the sources.

Following assumptions A1–A3, the covariance matrix of y(t) is given by

$$\boldsymbol{R} = \mathbb{E}[\boldsymbol{y}(t)\boldsymbol{y}^{H}(t)] = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{P}\boldsymbol{A}^{H}(\boldsymbol{\theta}) + \sigma_{n}^{2}\boldsymbol{I}, \qquad (3)$$

where $P = \text{diag}(p_1, p_2, \dots, p_K)$ is the covariance matrix of the source signals. Vectorizing R in (3) leads to

$$\boldsymbol{r} = (\boldsymbol{A}^* \odot \boldsymbol{A})\boldsymbol{p} + \sigma_n^2 \operatorname{vec}(\boldsymbol{I}), \qquad (4)$$

where $\boldsymbol{p} = [p_1, p_2, ..., p_k]^T$. It can be observed that \boldsymbol{r} resembles a single measurement vector whose steering matrix embeds a difference coarray whose sensor locations are given by $\mathcal{D}_{co} = \{(d_m - d_n, 0) | d_m, d_n \in \mathcal{D}\}$. The matrix $(\boldsymbol{A}^* \odot \boldsymbol{A})$ is the steering matrix of the difference coarray. The vector \boldsymbol{p} resembles a deterministic source signal, and the vector $\sigma_n^2 \operatorname{vec}(\boldsymbol{I})$ resembles a deterministic additive noise.

It has been shown that for well-designed sparse linear arrays, \mathcal{D}_{co} contains a ULA of $2M_{co}-1$ sensors centered at the origin, with $M_{co} > M$ [11]. Through redundancy averaging, we can construct a new measurement vector $\boldsymbol{z} = \boldsymbol{F}\boldsymbol{r}$ that resembles a measurement vector of this ULA, where $\boldsymbol{F} \in \mathbb{R}^{(2M_{co}-1)\times M^2}$. The precise definition of \boldsymbol{F} can be found in [16]. Fig. 1 provides an illustrative example of the relationship between the physical array, the difference coarray, and the central ULA part of the difference coarray [16].

Through spatial smoothing, we can construct an augmented covariance matrix \mathbf{R}_{ss} of the order $M_{co} \times M_{co}$, such that

$$\boldsymbol{R}_{\rm ss} = \frac{1}{M_{\rm co}} \sum_{l=1}^{M_{\rm co}} \boldsymbol{\Gamma}_l \boldsymbol{z} \boldsymbol{z}^H \boldsymbol{\Gamma}_l^T, \qquad (5)$$

where $\Gamma_l = [\mathbf{0}_{M_{co} \times (l-1)} I_{M_{co} \times M_{co}} \mathbf{0}_{M_{co} \times (M_{co}-l)}]$. By applying MUSIC to \mathbf{R}_{ss} , more sources than the number of sensors can be resolved. The MUSIC algorithm applied to \mathbf{R}_{ss} is referred as SS-MUSIC [11].

In the derivations above, we assume that each sensor is perfectly calibrated and placed at its nominal position, which may not be true in practice. To obtain a more general perturbation model, we consider sensor location errors along both the x-axis and the y-axis¹. We use $\boldsymbol{u} = [u_1, u_2, \dots, u_M]^T$ to denote the sensor locations errors along the x-axis, and $\boldsymbol{v} = [v_1, v_2, \dots, v_M]^T$ to denote the sensor location errors along the y-axis. The perturbed sensor locations are then given by $\tilde{\mathcal{D}} = \{(d_1 + u_1, v_1), (d_2 + u_2, v_2), \dots, (d_M + u_M, v_M)\}$. When the sensor location errors are large, the linear array structure will be completely destroyed, resulting large DOA estimation errors that are difficult to characterize. Therefore, our performance analysis will focus on cases when the sensor location errors are small. In addition to assumptions A1–A3, we make the following additional assumption:

A4 The sensor location errors are small compared with d_0 . Let $\boldsymbol{\delta} = [\boldsymbol{u}^T \ \boldsymbol{v}^T]^T$ denote the collection of sensor location error parameters. Under assumption A1–A4, the N snapshots received by the perturbed array can be expressed as

$$\tilde{\boldsymbol{y}}(t) = \tilde{\boldsymbol{A}}(\boldsymbol{\theta}, \boldsymbol{\delta})\boldsymbol{x}(t) + \boldsymbol{n}(t), t = 1, 2, \dots, N,$$
(6)

where $\tilde{A}(\theta, \delta)$ denotes the perturbed steering matrix. We name (6) the *deterministic error model*, reflecting the fact that sensor location errors do not change during the N snapshots.

One extension to the deterministic error model is the *stochastic error model*, where the sensor location errors are time-dependent. Such a model is applicable when the array is mounted on a non-stationary surface (e.g. [28], [29]), and the sensor location errors cannot be assumed constant during the N snapshots. By replacing u, v and δ with their time-dependent counterparts, we can express the N snapshots received by the perturbed array as

$$\tilde{\boldsymbol{y}}(t) = \tilde{\boldsymbol{A}}(\boldsymbol{\theta}, \boldsymbol{\delta}(t))\boldsymbol{x}(t) + \boldsymbol{n}(t), t = 1, 2, \dots, N.$$
(7)

For both models, the covariance matrix will deviate from the nominal one due to sensor location errors. If we neglect the sensor location errors and follow the same procedure of redundancy averaging and spatial smoothing as above, the resulting augmented covariance matrix will be a perturbed version of the nominal R_{ss} , which will degrade the DOA estimation performance. In the following sections, we will analyze in detail how such degradations are related to the sensor location errors.

It should be noted that there exists other coarray-based MUSIC algorithms, such as DA-MUSIC [30]. Because DA-MUSIC shares the same asymptotic MSE as SS-MUSIC [16], the results in the following sections are also applicable to DA-MUSIC. For other coarray-based MUSIC algorithms, our results are not directly applicable because the augmented covariance matrix is constructed differently. For these algorithms, we can view the construction process of the augmented covariance matrix as a function that maps the original covariance matrix to the augmented covariance matrix. If such a function is analytic, we can obtain its Taylor series and analyze the effect of the sensor location errors by following the ideas in [16] and the following sections. However, the resulting expressions may be more involved. To avoid complications, in the following sections, we will focus on analyzing how sensor location errors impact the DOA estimation performance of SS-MUSIC.

III. THE DETERMINISTIC ERROR MODEL

In the deterministic error model, the perturbed covariance matrix is given by

$$\tilde{\boldsymbol{R}} = \tilde{\boldsymbol{A}}(\boldsymbol{\theta}, \boldsymbol{\delta}) \boldsymbol{P} \tilde{\boldsymbol{A}}^{H}(\boldsymbol{\theta}, \boldsymbol{\delta}) + \sigma_{n}^{2} \boldsymbol{I},$$
(8)

where

$$\tilde{A}_{ik} = \exp\left[j\frac{2\pi}{\lambda}(d_i\sin\theta_k + u_i\sin\theta_k + v_i\cos\theta_k)\right].$$

The corresponding observation model of the difference coarray is then given by

$$\tilde{\boldsymbol{r}} = (\tilde{\boldsymbol{A}}^* \odot \tilde{\boldsymbol{A}})\boldsymbol{p} + \sigma_n^2 \operatorname{vec}(\boldsymbol{I}).$$
(9)

Here we drop the explicit dependencies on θ , δ for notational simplicity. The matrix $(\tilde{A}^* \odot \tilde{A})$ now resembles a steering matrix of the perturbed difference coarray, whose sensor locations are given by $\tilde{\mathcal{D}}_{co} = \{(d_m - d_n + u_m - u_n, v_m - v_n) | m, n = 1, 2, \ldots, M\}$. As illustrated in Fig. 2, the perturbed difference coarray no longer embeds a ULA, and can no longer be divided into multiple overlapping subarrays of the same shape. Consequently, applying SS-MUSIC to the perturbed difference coarray model without error compensations will lead to degraded DOA estimation performance.



Fig. 2. Illustration of a perturbed difference coarray: (a) a co-prime array and its difference coarray; (b) a perturbed co-prime array and its perturbed difference coarray.

To establish the link between the coarray perturbation and the DOA estimation errors, we start with the perturbed steering matrix \tilde{A} . Because \tilde{A} is analytic in the neighborhood of $\delta = 0$, we can linearize \tilde{A} around $\delta = 0$ via the first-order Taylor expansion under assumption A4:

$$\tilde{A} = A + U\tilde{A}_u + V\tilde{A}_v + o(\delta), \qquad (10)$$

¹We do not need to consider the perturbations along the *z*-axis under the far-field and co-planar assumption of the source signals.

where

$$\boldsymbol{U} = \operatorname{diag}(u_1, u_2, \dots, u_M), \tag{11a}$$

$$\boldsymbol{V} = \operatorname{diag}(v_1, v_2, \dots, v_M), \tag{11b}$$

$$\tilde{\boldsymbol{A}}_{u} = j \frac{2\pi}{\lambda} \boldsymbol{A} \boldsymbol{D}_{\mathrm{s}},\tag{11c}$$

$$ilde{m{A}}_v = j rac{2\pi}{\lambda} m{A} m{D}_{
m c}, ext{(11d)}$$

$$\boldsymbol{D}_{\mathrm{s}} = \mathrm{diag}(\sin\theta_1, \sin\theta_2, \dots, \sin\theta_K),$$
 (11e)

$$\boldsymbol{D}_{\mathrm{c}} = \mathrm{diag}(\cos\theta_1, \cos\theta_2, \dots, \cos\theta_K),$$
 (11f)

and $o(\delta)$ denotes the higher order terms with respect to δ . The perturbed covariance matrix \tilde{R} can then be approximated as

$$\tilde{\boldsymbol{R}} = \boldsymbol{R} + \boldsymbol{U}\tilde{\boldsymbol{A}}_{u}\boldsymbol{P}\boldsymbol{A}^{H} + \boldsymbol{A}\boldsymbol{P}\tilde{\boldsymbol{A}}_{u}^{H}\boldsymbol{U} + V\tilde{\boldsymbol{A}}_{v}\boldsymbol{P}\boldsymbol{A}^{H} + \boldsymbol{A}\boldsymbol{P}\tilde{\boldsymbol{A}}_{v}^{H}\boldsymbol{V} + o(\boldsymbol{\delta}).$$
(12)

In practice, the true covariance matrix is unknown, and we obtain only the estimate of $\tilde{\boldsymbol{R}}$ with $\hat{\boldsymbol{R}} = \frac{1}{N} \sum_{t=1}^{N} \boldsymbol{y}(t) \boldsymbol{y}(t)^{H}$. Hence, the discrepancy between the estimate, $\hat{\boldsymbol{R}}$, and nominal covariance matrix, \boldsymbol{R} , can be decomposed into two parts:

$$\Delta \boldsymbol{R} = \hat{\boldsymbol{R}} - \boldsymbol{R} = \underbrace{(\hat{\boldsymbol{R}} - \tilde{\boldsymbol{R}})}_{\boldsymbol{E}} + \underbrace{(\tilde{\boldsymbol{R}} - \boldsymbol{R})}_{\boldsymbol{G}}, \quad (13)$$

where E denotes the estimation errors resulting from finite snapshots, and G denotes the estimation errors resulting from sensor location errors. To derive the asymptotic MSE expression of SS-MUSIC in the presence of sensor location errors, we make use of the following theorem [16].

Theorem 1. Let $\hat{\theta}_k^{(SS)}$ be the estimated value of the k-th DOA by SS-MUSIC, and $\Delta r = \text{vec}(\Delta R)$. Denote the number of virtual sensors in the central ULA part of the difference coarray \mathcal{D}_{co} by $2M_{co} - 1$. If ΔR is Hermitian, then for sufficiently small Δr , the DOA estimator error of SS-MUSIC is given by

$$\Delta \theta_k := \hat{\theta}_k^{(\mathrm{SS})} - \theta_k = -(\gamma_k p_k)^{-1} \Re(\boldsymbol{\xi}_k^T \Delta \boldsymbol{r}) + o(\|\Delta \boldsymbol{r}\|),$$
(14)

where

$$\boldsymbol{\xi}_k = \boldsymbol{F}^T \boldsymbol{\Gamma}^T (\boldsymbol{\beta}_k \otimes \boldsymbol{\alpha}_k), \qquad (15a)$$

$$\boldsymbol{\alpha}_k^T = -\boldsymbol{e}_k^T \boldsymbol{A}_{\rm co}^\dagger, \tag{15b}$$

$$\boldsymbol{\beta}_{k} = \boldsymbol{\Pi}_{\boldsymbol{A}_{\rm co}}^{\perp} \dot{\boldsymbol{a}}_{\rm co}(\boldsymbol{\theta}_{k}), \qquad (15c)$$

$$\boldsymbol{\gamma}_{k} = \dot{\boldsymbol{a}}_{\mathrm{co}}(\theta_{k})^{H} \boldsymbol{\Pi}_{\boldsymbol{A}_{\mathrm{co}}}^{\perp} \dot{\boldsymbol{a}}_{\mathrm{co}}(\theta_{k}), \qquad (15\mathrm{d})$$

$$\boldsymbol{\Gamma} = [\boldsymbol{\Gamma}_{M_{\text{co}}}^T \ \boldsymbol{\Gamma}_{M_{\text{co}}-1}^T \ \cdots \ \boldsymbol{\Gamma}_1^T]^T, \quad (15e)$$

$$\dot{\boldsymbol{a}}_{\rm co}(\theta_k) = \frac{\partial \boldsymbol{a}_{\rm co}(\theta_k)}{\partial \theta_l}.$$
(15f)

Here \mathbf{A}_{co} is the steering matrix of a M_{co} -sensor ULA whose sensor locations are given by $\{0, d_0, \ldots, (M_{co} - 1)d_0\}$, $\Gamma_i = [\mathbf{0}_{M_{co} \times (i-1)} \mathbf{I}_{M_{co} \times M_{co}} \mathbf{0}_{M_{co} \times (M_{co} - i)}]$ follows the same definition as in (5), and \mathbf{F} follows the same definition as in [16, Appendix A].

It is straightforward to verify that R is still Hermitian in the presence of sensor location errors. Combining (13) and Theorem 1 and neglecting all the high order terms, we obtain

$$\Delta \theta_k \doteq -(\gamma_k p_k)^{-1} \Re[\boldsymbol{\xi}_k^T(\boldsymbol{e} + \boldsymbol{g})], \qquad (16)$$

where \doteq denotes equality up to the first order, e = vec(E), and g = vec(G). Hence, for a large number of snapshots, the asymptotic MSE can be evaluated as

$$\mathbb{E}[\Delta \theta_k^2] \doteq \frac{\mathbb{E}\{[\Re(\boldsymbol{\xi}_k^T(\boldsymbol{e} + \boldsymbol{g}))]^2\}}{\gamma_k^2 p_k^2}.$$
(17)

Using the fact that $\Re(AB) = \Re(A)\Re(B) - \Im(A)\Im(B)$, we can expand the numerator in (17) as follows:

$$\mathbb{E}\{[\Re(\boldsymbol{\xi}_{k}^{T}(\boldsymbol{e}+\boldsymbol{g}))]^{2}\}$$

= $\Re(\boldsymbol{\xi}_{k})^{T}\mathbb{E}[\Re(\boldsymbol{e}+\boldsymbol{g})\Re(\boldsymbol{e}+\boldsymbol{g})^{T}]\Re(\boldsymbol{\xi}_{k})$
+ $\Im(\boldsymbol{\xi}_{k})^{T}\mathbb{E}[\Im(\boldsymbol{e}+\boldsymbol{g})\Im(\boldsymbol{e}+\boldsymbol{g})^{T}]\Im(\boldsymbol{\xi}_{k})$
- $2\Re(\boldsymbol{\xi}_{k})^{T}\mathbb{E}[\Re(\boldsymbol{e}+\boldsymbol{g})\Im(\boldsymbol{e}+\boldsymbol{g})^{T}]\Im(\boldsymbol{\xi}_{k}).$ (18)

Because $\mathbb{E}[e] = 0$, we have

$$\begin{split} \mathbb{E}[\Re(\boldsymbol{e}+\boldsymbol{g})\Re(\boldsymbol{e}+\boldsymbol{g})^T] &= \mathbb{E}[\Re(\boldsymbol{e})\Re(\boldsymbol{e})^T] + \Re(\boldsymbol{g})\Re(\boldsymbol{g})^T, \\ \mathbb{E}[\Im(\boldsymbol{e}+\boldsymbol{g})\Im(\boldsymbol{e}+\boldsymbol{g})^T] &= \mathbb{E}[\Im(\boldsymbol{e})\Im(\boldsymbol{e})^T] + \Im(\boldsymbol{g})\Im(\boldsymbol{g})^T, \\ \mathbb{E}[\Re(\boldsymbol{e}+\boldsymbol{g})\Im(\boldsymbol{e}+\boldsymbol{g})^T] &= \mathbb{E}[\Re(\boldsymbol{e})\Im(\boldsymbol{e})^T] + \Re(\boldsymbol{g})\Im(\boldsymbol{g})^T. \end{split}$$

Hence we can expand (18) as (19). The first three terms evaluate into $\Re[\boldsymbol{\xi}_k^H(\tilde{\boldsymbol{R}} \otimes \tilde{\boldsymbol{R}}^T)\boldsymbol{\xi}_k]/N$. The derivation follows the same idea as in [16, Appendix C], but with \boldsymbol{R} replaced with $\tilde{\boldsymbol{R}}$. The second three terms can be combined into $\Re(\boldsymbol{g}^T\boldsymbol{\xi}_k)^T\Re(\boldsymbol{g}^T\boldsymbol{\xi}_k)$. To obtain the final MSE expression, we still need to expand \boldsymbol{g} in terms of $\boldsymbol{\delta}$, which requires Lemma 1.

Lemma 1. Let D = diag(d) be a diagonal matrix. Then $\text{vec}(DX) = (X^T \odot I)d$ and $\text{vec}(XD) = (I \odot X)d$ for any matrix X with a proper shape.

Proof. The two equalities follow immediately from the following fact [31]: for any diagonal matrix X and any two matrices A, B with proper shapes,

$$\operatorname{vec}(\boldsymbol{A}\boldsymbol{X}\boldsymbol{B}) = (\boldsymbol{B}^T \odot \boldsymbol{A})\operatorname{diag}(\boldsymbol{X}).$$
 (20)

Using Lemma 1 and (12), we can rewrite g as $B\delta + o(\delta)$, where $B = [B_u \ B_v]$ and

$$\boldsymbol{B}_{\boldsymbol{u}} = \boldsymbol{I} \odot (\boldsymbol{A} \boldsymbol{P} \tilde{\boldsymbol{A}}_{u}^{H}) + (\boldsymbol{A} \boldsymbol{P} \tilde{\boldsymbol{A}}_{u}^{H})^{*} \odot \boldsymbol{I}, \qquad (21a)$$

$$\boldsymbol{B}_{\boldsymbol{v}} = \boldsymbol{I} \odot (\boldsymbol{A} \boldsymbol{P} \tilde{\boldsymbol{A}}_{v}^{H}) + (\boldsymbol{A} \boldsymbol{P} \tilde{\boldsymbol{A}}_{v}^{H})^{*} \odot \boldsymbol{I}.$$
(21b)

Substituting the expression for g back into (19), we obtain the following result.

Corollary 1. Under the deterministic error model, the asymptotic MSE of SS-MUSIC for the k-th DOA in the presence of small sensor location errors is given by

$$\frac{1}{p_k^2 \gamma_k^2} \left\{ \frac{1}{N} \Re[\boldsymbol{\xi}_k^H (\tilde{\boldsymbol{R}} \otimes \tilde{\boldsymbol{R}}^T) \boldsymbol{\xi}_k] + [\boldsymbol{\delta}^T \Re(\boldsymbol{B}^T \boldsymbol{\xi}_k)]_2^2 \right\}, \quad (22)$$

where $\boldsymbol{\xi}_k$ and \boldsymbol{B} follow the same definition in Theorem 1 and (21a)–(21b).

The asymptotic MSE (22) consists of two terms. The first term results from the estimation errors of the covariance matrix, which will vanish as the number of snapshots goes to infinity. It should be also noted that this term is also affected by the sensor location errors, because \tilde{R} depends on δ . However,

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$$\mathbb{E}\{[\Re(\boldsymbol{\xi}_{k}^{T}(\boldsymbol{e}+\boldsymbol{g}))]^{2}\} = \Re(\boldsymbol{\xi}_{k})^{T} \mathbb{E}[\Re(\boldsymbol{e})\Re(\boldsymbol{e})^{T}]\Re(\boldsymbol{\xi}_{k}) + \Im(\boldsymbol{\xi}_{k})^{T} \mathbb{E}[\Im(\boldsymbol{e})\Im(\boldsymbol{e})^{T}]\Im(\boldsymbol{\xi}_{k}) - 2\Re(\boldsymbol{\xi}_{k})^{T} \mathbb{E}[\Re(\boldsymbol{e})\Im(\boldsymbol{e})^{T}]\Im(\boldsymbol{\xi}_{k}) \\ + \Re(\boldsymbol{\xi}_{k})^{T}\Re(\boldsymbol{g})\Re(\boldsymbol{g})^{T}\Re(\boldsymbol{\xi}_{k}) + \Im(\boldsymbol{\xi}_{k})^{T}\Im(\boldsymbol{g})\Im(\boldsymbol{g})^{T}\Im(\boldsymbol{\xi}_{k}) - 2\Re(\boldsymbol{\xi}_{k})^{T}\Re(\boldsymbol{g})\Im(\boldsymbol{g})^{T}\Im(\boldsymbol{\xi}_{k}) \\ = \Re[\boldsymbol{\xi}_{k}^{H}(\tilde{\boldsymbol{R}}\otimes\tilde{\boldsymbol{R}}^{T})\boldsymbol{\xi}_{k}]/N + \Re(\boldsymbol{g}^{T}\boldsymbol{\xi}_{k})^{T}\Re(\boldsymbol{g}^{T}\boldsymbol{\xi}_{k})$$
(19)

given a sufficient number of snapshots N, such an effect is negligible after being divided by N. The second term is the result from sensor location errors, which will not vanish as the number of snapshots goes to infinity, leading to a constant bias among the DOA estimates.

Corollary 1 gives the asymptotic MSE for a particular realization of the sensor locations errors, δ . We are also interested in the ensemble behavior of (22) under different realizations of sensor location errors. Following the idea of the hybrid CRB, we assume that the sensor location errors δ follows a Gaussian prior $\mathcal{N}(\mathbf{0}, \mathbf{C})$ [32], and evaluate the average asymptotic MSE under this Gaussian prior. The results are summarized in Corollary 2.

Corollary 2. Let $\delta \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$, where $\|\mathbf{C}\|$ is sufficiently small such that the high order moments of δ/d_0 are $o(\|\mathbf{C}\|)$. Then then the average asymptotic MSE (AAMSE) of SS-MUSIC in the presence of sensor location errors is given by

$$\frac{1}{p_k^2 \gamma_k^2} \left\{ \frac{1}{N} \Re[\boldsymbol{\xi}_k^H (\boldsymbol{R} \otimes \boldsymbol{R}^T) \boldsymbol{\xi}_k] + \Re(\boldsymbol{B}^T \boldsymbol{\xi}_k)^T \boldsymbol{C} \Re(\boldsymbol{B}^T \boldsymbol{\xi}_k) \right\},$$
(23)

Proof. Let $\Delta = U\tilde{A}_u P A^H + A P \tilde{A}_u^H U + V \tilde{A}_v P A^H + A P \tilde{A}_v^H V$. Using (12), we have

$$\tilde{\boldsymbol{R}} \otimes \tilde{\boldsymbol{R}}^T = \boldsymbol{R} \otimes \boldsymbol{R}^T + \boldsymbol{R} \otimes \boldsymbol{\Delta}^T + \boldsymbol{\Delta} \otimes \boldsymbol{R}^T + o(\|\boldsymbol{C}\|).$$

Because $\mathbb{E}_{\delta}[\Delta] = 0$, using the assumption that the high order moments of δ/d_0 are o(||C||), we obtain $\mathbb{E}_{\delta}[\tilde{R} \otimes \tilde{R}^T] \doteq R \otimes$ R^T . This leads to the first term in (23). The second term in (23) is due to the fact that $\mathbb{E}_{\delta}[\delta\delta^T] = C$. The remaining high order terms are still o(||C||) under the assumption that that the high order moments of δ/d_0 are o(||C||).

Because the second error term in (23) is linear in C, we can use $\Re(B^T \xi_k)^T \Re(B^T \xi_k)$ as a sensitivity metric of the robustness of SS-MUSIC against the sensor location errors for the *k*-th DOA. It can be observed that this term is affected by both the physical array geometry and the coarray geometry. The physical array geometry is encoded in the matrix B, which depends on the nominal physical array steering matrix A. The coarray geometry is encoded in the vector ξ_k , which depends on the coarray steering matrix A_{co} as well as the transform matrix F. This observation implies that even if two sparse linear arrays share the same coarray structure, their sensitivities against model errors may not be the same.

Corollary 3. Assume all sources share the same power p. Let $\varepsilon(\theta_k)$ denote the AAMSE of the k-th DOA in Corollary 2. Fixing σ_n^2 , we have

$$\lim_{p \to \infty} \varepsilon(\theta_k) = \frac{1}{\gamma_k^2} \left\{ \frac{1}{N} \| \boldsymbol{\xi}_k^H (\boldsymbol{A} \otimes \boldsymbol{A}^*) \|_2^2 + \Re(\bar{\boldsymbol{B}}^T \boldsymbol{\xi}_k)^T \boldsymbol{C} \Re(\bar{\boldsymbol{B}}^T \boldsymbol{\xi}_k) \right\},$$
(24)

where $\bar{B} = [\bar{B}_u \ \bar{B}_v]$, and

$$\begin{split} \bar{\boldsymbol{B}}_{\boldsymbol{u}} &= \boldsymbol{I} \odot (\boldsymbol{A} \tilde{\boldsymbol{A}}_{u}^{H}) + (\boldsymbol{A} \tilde{\boldsymbol{A}}_{u}^{H})^{*} \odot \boldsymbol{I}, \\ \bar{\boldsymbol{B}}_{\boldsymbol{v}} &= \boldsymbol{I} \odot (\boldsymbol{A} \tilde{\boldsymbol{A}}_{v}^{H}) + (\boldsymbol{A} \tilde{\boldsymbol{A}}_{v}^{H})^{*} \odot \boldsymbol{I}. \end{split}$$

Proof. The result follows directly from Corollary 2 and [16, Corollary 1]. \Box

The first term in (24) is the limiting expression of the asymptotic MSE of SS-MUSIC in the absence of sensor location errors as the SNR approaches infinity, which is generally non-zero when multiple sources are present [16]. The second term in (24) is the result from sensor location errors. Because \bar{B} is independent of the source power p, we conclude that the DOA estimation bias of SS-MUSIC introduced by the sensor location errors cannot be mitigated by increasing the SNR alone.

IV. EXTENSION: THE STOCHASTIC ERROR MODEL

In the stochastic error model, we assume that the location errors vary in each snapshot, such that the t-th snapshot is given by

$$\tilde{\boldsymbol{y}}(t) = \tilde{\boldsymbol{A}}(\boldsymbol{\delta}(t))\boldsymbol{s}(t) + \boldsymbol{n}(t), \qquad (25)$$

in which $\delta(t)$ follows some stochastic model. To avoid complications and obtain a general idea of the impact of stochastic sensor location errors, we make the following additional assumption:

A5 The sensor location errors $\delta(t)$ are i.i.d. and are uncorrelated from both the source signals s(t) and the additive noise n(t).

Because $\hat{A}(\delta(t))$ is nonlinear in the random variable $\delta(t)$, \tilde{y} no longer follows the complex circularly-symmetric Gaussian distribution as in the deterministic error model. Consequently, it is rather difficult to derive the distribution of \hat{R} for the stochastic error model in the case of a finite number of snapshots. On the other hand, as implied by (13), the effect of sensor location errors dominates only when the number of snapshots is sufficiently large. Hence for the stochastic error model, we will analyze how the sensor location errors affect the estimation performance when an infinite number of snapshots is available.

Under assumption A1–A4, the perturbed covariance matrix can be evaluated as

$$\begin{split} \tilde{\boldsymbol{R}} &= \mathbb{E}[\boldsymbol{y}(t)\boldsymbol{y}^{H}(t)] \\ &= \mathbb{E}[\tilde{\boldsymbol{A}}(\boldsymbol{\delta}(t))\boldsymbol{s}(t)\boldsymbol{s}^{H}(t)\tilde{\boldsymbol{A}}^{H}(\boldsymbol{\delta}(t))] + \mathbb{E}[\tilde{\boldsymbol{A}}(\boldsymbol{\delta}(t))\boldsymbol{s}(t)\boldsymbol{n}^{H}(t)] \\ &+ \mathbb{E}[\boldsymbol{n}(t)\boldsymbol{s}^{H}(t)\tilde{\boldsymbol{A}}^{H}(\boldsymbol{\delta}(t))] + \mathbb{E}[\boldsymbol{n}(t)\boldsymbol{n}^{H}(t)]. \\ &= \underbrace{\mathbb{E}[\tilde{\boldsymbol{A}}(\boldsymbol{\delta}(t))\boldsymbol{s}(t)\boldsymbol{s}^{H}(t)\tilde{\boldsymbol{A}}^{H}(\boldsymbol{\delta}(t))]}_{S} + \sigma_{n}^{2}\boldsymbol{I}, \end{split}$$

where the cross terms vanish, because the sources and the additive noise have zero means and are uncorrelated. The first term S can be expressed as

$$\boldsymbol{S} = \sum_{i=1}^{K} \sum_{l=1}^{K} \mathbb{E}[\tilde{\boldsymbol{a}}(\theta_i, \boldsymbol{\delta}(t)) s_i(t) s_l^*(t) \tilde{\boldsymbol{a}}^H(\theta_l, \boldsymbol{\delta}(t))],$$

whose (m, n)-th element is given by

$$S_{mn} = \sum_{i=1}^{K} \sum_{l=1}^{K} \mathbb{E}[\tilde{a}_m(\theta_i, \boldsymbol{\delta}(t)) \tilde{a}_n^*(\theta_l, \boldsymbol{\delta}(t)) s_i(t) s_l^*(t)].$$
(26)

Using assumption A5, we can decouple the expectation evaluations with respect to $\delta(t)$ and s(t). Noting that $\mathbb{E}[s_i(t)s_l^*(t)] = p_l$ only if i = l, and is otherwise 0, we need to consider only the terms where i = l. We can then rewrite (26) as

$$S_{mn} = \sum_{k=1}^{K} p_i \mathbb{E}[\tilde{a}_m(\theta_k, \boldsymbol{\delta}(t)) \tilde{a}_n^*(\theta_k, \boldsymbol{\delta}(t))]$$

$$= \sum_{k=1}^{K} p_i a_m(\theta_k) a_n^*(\theta_k) \mathbb{E}\left\{e^{j(\boldsymbol{t}_{k,m} - \boldsymbol{t}_{k,n})^T \boldsymbol{\delta}}\right\} \qquad (27)$$

$$= \sum_{k=1}^{K} p_i a_m(\theta_k) a_n^*(\theta_k) \phi_{\boldsymbol{\delta}}(\boldsymbol{t}_{k,m} - \boldsymbol{t}_{k,n}),$$

where $\phi_{\delta}(t)$ is the characteristic function of $\delta(t)$, $t_{k,n} = \frac{2\pi}{\lambda} \begin{bmatrix} e_M^{(n)} \sin \theta_k \\ e_M^{(n)} \cos \theta_k \end{bmatrix}$, and $e_M^{(n)}$ is an *M*-dimensional vector with only the *n*-th element being one and other elements being zero. Let Φ_k be an $M \times M$ matrix whose (m, n)-th element is given by $\phi_{\delta}(t_{k,m} - t_{k,n})$. We can then express \tilde{R} as

$$\tilde{\boldsymbol{R}} = \sum_{k=1}^{K} p_k [\boldsymbol{a}(\theta_k) \boldsymbol{a}^H(\theta_k)] \circ \boldsymbol{\Phi}_k + \sigma_n^2 \boldsymbol{I}.$$
 (28)

Here, the effect of the sensor location errors is encoded in matrices Φ_k . Because $t_{k,m}$ depends on the *k*-th DOA, the effect of sensor location errors is generally DOA dependent and cannot be treated as colored Gaussian noise.

Vectorizing the (28) leads to

$$\tilde{\boldsymbol{r}} = [(\boldsymbol{A}^* \odot \boldsymbol{A}) \circ \boldsymbol{\Phi}] \boldsymbol{p} + \sigma_n^2 \operatorname{vec}(\boldsymbol{I}), \quad (29)$$

where $\Phi = [\operatorname{vec}(\Phi_1) \operatorname{vec}(\Phi_2) \cdots \operatorname{vec}(\Phi_K)]$. Comparing (29) with (4), we observe that, under the stochastic error model, the coarray steering matrix $(A^* \odot A)$ is modulated by Φ . Because characteristic functions usually do not evaluate to one outside the origin, Φ will not be a matrix of ones and the corresponding difference coarray model will be perturbed.

To give a better idea of (28) and (29), we consider the case when $\delta(t)$ follows a zero-mean Gaussian distribution with the covariance matrix denoted by C. We partition C as $\begin{bmatrix} C_{uu} & C_{uv} \\ C_{vu} & C_{vv} \end{bmatrix}$, where C_{uu} and C_{vv} are the covariance of the location errors along the x-axis and y-axis, respectively, and C_{uv} denotes the corresponding cross covariance. The corresponding characteristic function of $\delta(t)$ is then given by $\phi_{\delta}(t) = \exp(-1/2t^T Ct)$. Substituting $t_{k,n}$ into $\phi_{\delta}(t)$ and

expanding the terms in the exponent, we obtain that in the Gaussian case

$$\Phi_k(m,n) = \exp\left\{-\frac{2\pi^2}{\lambda^2}[\mu_1(m,n)\sin^2\theta_k + \mu_2(m,n)\cos^2\theta_k + 2\mu_3(m,n)\sin\theta_k\cos\theta_k]\right\},$$
(30)

where

$$\mu_1(m,n) = C_{uu}(m,m) + C_{uu}(n,n) - 2C_{uu}(m,n),$$

$$\mu_2(m,n) = C_{vv}(m,m) + C_{vv}(n,n) - 2C_{vv}(m,n),$$

$$\mu_3(m,n) = C_{uv}(m,m) + C_{uv}(n,n) - C_{uv}(m,n) - C_{uv}(n,m)$$

We also observe that $\Phi_k(m,n)$ is still dependent on the *k*-th DOA. Hence for a general covariance matrix, the effect of the random sensor location errors is still DOA dependent. However, as shown in the following proposition, for certain covariance matrices, $\Phi_k(m, n)$ is independent of *k*.

Proposition 1. Let $\delta(t) \sim \mathcal{N}(0, C)$. Then Φ_k (k = 1, 2, ..., K) are independent of the DOAs if and only if $\mu_1(m, n) = \mu_2(m, n)$ and $\mu_3(m, n) = 0$ holds for every m, n = 1, 2, ..., M.

Proof. Let $a, b, c \in \mathbb{C}$. Define $f(\theta) = a \sin^2 \theta + b \cos^2 \theta + c \sin \theta \cos \theta$. It suffices to show that $f(\theta)$ is a constant for all $\theta \in (-\pi/2, \pi/2)$ if and only if a = b and c = 0.

The sufficiency is trivial and we need to show only necessity. Suppose $f(\theta) = d, \forall \theta \in (-\pi/2, \pi/2)$. Choose $\theta = \pi/4$ and we obtain a + b + c = 2d. Choose $\theta = -\pi/4$ and we obtain a + b - c = 2d. Therefore c must be 0. Choose $\theta = 0$ and we obtain b = d, which implies that $(a - b)\sin^2 \theta = 0$ must hold for every $\theta \in (-\pi/2, \pi/2)$. Therefore we must have a = b.

One special case that satisfies the conditions given in Proposition 1 is when $C = \sigma_p^2 I$, which leads to the following corollary.

Corollary 4. Let $\delta_t \sim \mathcal{N}(\mathbf{0}, \sigma_p^2 \mathbf{I})$. Then

$$\tilde{\boldsymbol{R}} = C_1 \Big\{ \boldsymbol{A} \boldsymbol{P} \boldsymbol{A}^H + \frac{1}{C_1} \Big[\sigma_n^2 + (1 - C_1) \sum_{k=1}^K p_k \Big] \boldsymbol{I} \Big\}, \quad (31)$$

where $C_1 = \exp(-4\pi^2 \sigma_{\rm p}^2 / \lambda^2)$.

Proof. The expression (31) can be obtained by substituting $C = \sigma_p^2 I$ into (30) and simplifying the resulting \tilde{R} according to (28).

We observe that if the sensor location perturbations are i.i.d. zero-mean Gaussian with the same variance, the effect of the sensor location errors can be indeed modeled as additive white noise as the number of snapshots goes to infinity. The signal subspace remains unchanged. However, the effective SNR is decreased because $0 < C_1 < 1$. In this special case, we can approximate the asymptotic MSE of SS-MUSIC for the *k*-th DOA with $\Re[\boldsymbol{\xi}_k^H(\boldsymbol{R} \otimes \boldsymbol{R}^T)\boldsymbol{\xi}_k]/(N\gamma_k^2 p_k^2)$, but with the original noise variance σ_n^2 replaced with the "effective noise variance"

$$\frac{1}{C_1} \Big[\sigma_n^2 + (1 - C_1) \sum_{k=1}^K p_k \Big].$$

V. THE CRAMÉR RAO BOUND

The CRB gives the lower bound on the minimum variance an unbiased estimator can achieve. In this section, we derive the CRB for general sparse linear arrays under the deterministic error model. In addition to the DOAs, source powers, and noise power, we also treat sensor location errors as unknown parameters. To obtain a more general expression of the FIM, we assume that the precise sensor locations are partially known. This assumption includes the case when sensor location errors among all the sensors are unknown. Let $\{i_1, i_2, \ldots, i_{M_1}\} \subseteq \{1, 2, \ldots, M\}$ denote the indices of sensors with unknown location errors along the x-axis, and $\{l_1, l_2, \ldots, l_{M_2}\} \subseteq \{1, 2, \ldots, M\}$ denote the indices of sensors with unknown location errors along the y-axis. The collection of unknown parameters is given by the $(2K+M_1+M_2+1) \times 1$ real vector:

$$\boldsymbol{\eta} = [\boldsymbol{\theta}^T, \boldsymbol{p}^T, u_{i_1}, \cdots, u_{i_{M_1}}, v_{l_1}, \cdots, v_{l_{M_2}}, \sigma_n^2]^T.$$
(32)

The FIM is then given by:

Proposition 2. Under assumptions A1–A3, the FIM of the deterministic error model is give by

$$\boldsymbol{J} = N\boldsymbol{M}^{H}(\tilde{\boldsymbol{R}}^{T} \otimes \tilde{\boldsymbol{R}})^{-1}\boldsymbol{M}.$$
(33)

Here,

$$M = \begin{bmatrix} \frac{\partial \tilde{r}}{\partial \theta} & \frac{\partial \tilde{r}}{\partial p} & \frac{\partial \tilde{r}}{\partial u} & \frac{\partial \tilde{r}}{\partial v} & \frac{\partial \tilde{r}}{\partial \sigma_{n}^{2}} \end{bmatrix},$$
(34)

where

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\theta}} = (\tilde{\boldsymbol{A}}_{\theta}^* \odot \tilde{\boldsymbol{A}} + \tilde{\boldsymbol{A}}^* \odot \tilde{\boldsymbol{A}}_{\theta}) \boldsymbol{P}, \qquad (35a)$$

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{p}} = \tilde{\boldsymbol{A}}^* \odot \tilde{\boldsymbol{A}},\tag{35b}$$

$$\frac{\partial \tilde{\boldsymbol{r}}}{\partial \boldsymbol{u}} = [(\tilde{\boldsymbol{A}}\boldsymbol{P}\tilde{\boldsymbol{A}}_{u}^{H})^{*}\boldsymbol{L}_{1}] \odot \boldsymbol{L}_{1} + \boldsymbol{L}_{1} \odot (\tilde{\boldsymbol{A}}\boldsymbol{P}\tilde{\boldsymbol{A}}_{u}^{H}\boldsymbol{L}_{1}), \quad (35c)$$

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{v}} = [(\tilde{\boldsymbol{A}}\boldsymbol{P}\tilde{\boldsymbol{A}}_{v}^{H})^{*}\boldsymbol{L}_{2}] \odot \boldsymbol{L}_{2} + \boldsymbol{L}_{2} \odot (\tilde{\boldsymbol{A}}\boldsymbol{P}\tilde{\boldsymbol{A}}_{v}^{H}\boldsymbol{L}_{2}), \quad (35d)$$

$$\frac{\partial \boldsymbol{r}}{\partial \sigma_{\rm n}^2} = \operatorname{vec}(\boldsymbol{I}_M), \tag{35e}$$

and

$$\begin{split} \boldsymbol{L}_{1} &= [\boldsymbol{e}_{M}^{(i_{1})} \ \boldsymbol{e}_{M}^{(i_{2})} \ \cdots \ \boldsymbol{e}_{M}^{(i_{M_{1}})}], \\ \boldsymbol{L}_{2} &= [\boldsymbol{e}_{M}^{(l_{1})} \ \boldsymbol{e}_{M}^{(l_{2})} \ \cdots \ \boldsymbol{e}_{M}^{(l_{M_{2}})}], \\ \tilde{\boldsymbol{A}}_{\theta} &= \left[\frac{\partial \tilde{\boldsymbol{a}}(\theta_{1})}{\partial \theta_{1}} \ \frac{\partial \tilde{\boldsymbol{a}}(\theta_{2})}{\partial \theta_{2}} \ \cdots \ \frac{\partial \tilde{\boldsymbol{a}}(\theta_{K})}{\partial \theta_{K}}\right]. \end{split}$$

Proof. See Appendix A.

If the FIM is nonsingular, the CRB for the DOAs can be readily obtained by inverting the FIM. However, this CRB does not always exist, due to the potential ambiguities introduced by sensor location errors. In the presence of sensor location errors, it is possible that certain combinations of DOAs, θ , and sensor location errors, δ , lead to the same perturbed steering matrix and same observations. Consequently, it is impossible to distinguished between these combinations from the observations. For a perturbed steering matrix, we formally define the local ambiguity as follows: **Definition 2.** An perturbed steering matrix $A(\theta, \delta)$ is called locally ambiguous if for any $(\theta, \delta) \in \Theta \times \Delta$, there exists a non-empty neighborhood $U \subset \Theta \times \Delta$, such that for any $(\tilde{\theta}, \tilde{\delta}) \in U$, $A(\tilde{\theta}, \tilde{\delta}) = A(\theta, \delta)$.

In practice, the first sensor is usually chosen as the reference sensor, whose location is assumed known. However, this is not sufficient to eliminate the local ambiguity, because the perturbed steering matrix remains the same if we rotate the array by a small angle and shift all the DOAs by the same amount. Even if we restrict the perturbation along the x-axis only, the local ambiguity still exists because we can obtain the same steering matrix by expanding or shrinking the whole array along the x-axis by a small amount and adjusting the DOAs accordingly. When such local ambiguities exist, the set of unknown parameters will be locally unidentifiable, leading to a singular Fisher information matrix (FIM) [33]. In the following discussion, we assume that the FIM is nonsingular.

Unlike the CRB derived in [32, Ch. 8], our CRB utilizes the assumption that the sources are uncorrelated. Observing that $(\tilde{R}^T \otimes \tilde{R})^{-1}$ is always full rank in the noisy case, the FIM is non-singular if and only if M is full rank. Because M is a matrix of dimension $M^2 \times (2K + M_1 + M_2 + 1)$, the FIM (33) can remain nonsingular for up to $O(M^2)$ sources. Therefore our CRB can work in the underdetermined case when K > M, while the CRB in [32] cannot. Our derivation is also different from that in [34]. In [34], the FIM is evaluated partition by partition under the assumption that both the source powers and the noise power are known. In our derivation, the FIM is derived in a "factorized" form, which is more concise than that in [34]. In addition, using our derivation, we conclude that the FIM can remain nonsingular for up to $O(M^2)$ sources. This conclusion is not easily seen from the derivation in [34].

Because the FIM (33) shares a form similar to the location error free FIM in [16], it is straightforward to show that the corresponding CRB depends on the SNRs instead of absolute values of p_k or σ_n^2 . For sparse linear arrays, we are particularly interested in the underdetermined case when $K \ge M$. In [16], we have shown that the location error free CRB remains positive definite even if the SNR approaches infinity. This unusual behavior still exists in the presence of sensor location errors. If both \tilde{A} and M are full rank, $\tilde{R}^T \otimes \tilde{R}$ remains full rank as σ_n^2 approaches 0, and the resulting FIM remains positive definite. Hence the Schur complement corresponding to the DOAs is also positive definite, leading to a positive definite CRB matrix. This behavior puts a strictly positive lower bound on the MSE of all unbiased estimators when $K \ge M$.

VI. NUMERICAL RESULTS

In this section, we use numerical simulations to demonstrate how sensor location errors affect the DOA estimation performance for sparse linear arrays. We consider both the deterministic error model and the stochastic error model. Unlike ULAs, sparse linear arrays sharing the same number of sensors can have different structures. For a comprehensive comparison, we consider two sets of sparse linear arrays throughout the simulations. The first set consists of four different sparse linear arrays sharing the same number of sensors:

- Co-prime (3,5): [0,3,5,6,9,10,12,15,20,25]d₀;
- MRA 10 [35]: [0, 1, 4, 10, 16, 22, 28, 30, 33, 35]d₀;
- Nested (4,6): [0,1,2,3,4,9,14,19,24,29]d₀;
- Nested (5,5): $[0, 1, 2, 3, 4, 5, 11, 17, 23, 29]d_0$.

The second set consists of four different sparse linear arrays sharing the same aperture:

- Co-prime (2,3): $[0,2,3,4,6,9]d_0$;
- MRA 5 [35]: [0,1,2,6,9]d₀;
- Nested (1,5): $[0,1,3,5,7,9]d_0$;
- Nested (4,2): $[0, 1, 2, 3, 4, 9]d_0$.

Throughout all experiments, we define the SNR as follows:

$$\mathrm{SNR} = 10 \log_{10} \frac{\min_{k=1,2,\dots,K} p_k}{\sigma_{\mathrm{n}}^2}$$

Given the results from L trials, we compute the empirical MSE with

$$MSE_{em} = \frac{1}{KL} \sum_{l=1}^{L} \sum_{k=1}^{K} (\hat{\theta}_{k}^{(l)} - \theta_{k}^{(l)})^{2},$$

where $\theta_k^{(l)}$ is the k-th DOA in the l-th trial, and $\hat{\theta}_k^{(l)}$ is the estimate of $\theta_k^{(l)}$.

A. Numerical Analysis of the Deterministic Error Model

We begin by verifying our closed-form asymptotic MSE expression (23) for the deterministic error model via numerical simulations. We consider 11 sources, which is more than the number of sensors, uniformly distributed between $-\pi/3$ and $\pi/3$ with equal power. We set the SNR to 0dB. We generate the sensor location errors from a zero-mean Gaussian distribution with covariance matrix $\sigma_p^2 I$. The magnitude of sensor location errors can then be tuned with $\sigma_{\rm p}^2$. We consider the first set of sparse linear arrays. We compute the difference between the AAMSE given by (23) and the empirical MSE under different combinations of snapshot numbers and magnitudes of perturbations. The results are summarized in Fig. 3. It can be observed that the empirical results agree very well with our analytical results when the number of snapshots is above 200 and the perturbation level is below 0.05. When the number of snapshots is small, the asymptotic assumption no longer holds, and the discrepancy between our analytical results and the empirical results becomes evident. When the magnitude of the sensor location errors is large, the high order terms with respect to the sensor location errors are no longer negligible, leading to discrepancies between our analytical results and the empirical results.

We next demonstrate how the DOA estimation errors vary with respect to sensor location errors for different types of sparse linear arrays. The results are plotted in Fig. 4 and Fig. 5. In Fig. 4, we plot the RMSE vs. σ_p/d_0 for four different sparse linear arrays with the same number of sensors. We observe that the MRA achieves the lowest RMSE, the coprime array achieves the highest RMSE, and the two nested arrays sit in the middle. This observation reflects the fact that the MRA has the largest aperture among the four arrays, while the co-prime array has the smallest. In Fig. 5, we plot the RMSE vs. $\sigma_{\rm p}/d_0$ for four different sparse linear arrays with the same aperture. We observe that while all four arrays show similar performance, MRA 5 is least sensitive to sensor location errors. Another interesting observation is that, Nested (1,5), Nested (4,2), and MRA 5, despite sharing the same central ULA part in their difference coarrays, show different sensitivities with respect the sensor location errors. This observation agrees with our analysis of (23).

Finally, we show how the variance of sensor location errors, $\sigma_{\rm p}$, affects the MSE of SS-MUSIC in high SNR regions. We consider 6 sources evenly placed between $-\pi/3$ and $\pi/3$, and fix the number of snapshots to 5000. Fig. 6 plots the results for Co-prime (3,5). We observe that the empirical MSEs well agree with our theoretical results. In the absence of sensor location errors, the MSE of SS-MUSIC converges to a positive constant as the SNR approaches infinity, which agrees with our analysis of Corollary 3. As the variance of sensor location errors increase, this positive constant also increases, because the bias resulting from sensor location errors grows larger. Additionally, we observe that the gap between the MSE values when the sensor location errors are present and when they are not present does not decrease as the SNR increases. This observation confirms our analysis of (23) that the bias cannot be mitigated by increasing only the SNR.

B. Numerical Analysis of the Stochastic Error Model

In this subsection, we verify our derivations in Section IV via numerical simulations. For the first set of sparse linear arrays, we consider 11 sources evenly distributed between $-\pi/3$ and $\pi/3$. For the second set of sparse linear arrays, we consider 6 sources evenly distributed between $-\pi/3$ and $\pi/3$. For both sets of sparse linear arrays, the number of sources is chosen to be larger than or equal to the number of sensors. We sample the sensor location errors $\delta(t)$ from a zero-mean Gaussian distribution with covariance matrix $\sigma_{\rm p}^2 I$, and the standard deviation of sensor location errors, $\sigma_{\rm p}$, is fixed to $0.1d_0$. Because the sensor location errors are i.i.d. zero-mean Gaussian, we approximate the analytical MSE by evaluating the location error free asymptotic MSE of SS-MUSIC [16] with the noise power replaced with the "effective noise power" given by Corollary 4. We fix the SNR to 0dB and vary the number of snapshots.

The results are plotted in Fig. 7 and Fig. 8. We observe that, when the number of snapshots is small, the empirical MSE deviates from the analytical MSE. As the number of snapshots increases, the empirical MSE approaches the our analytical approximation. This is because our analytical approximation is based on the assumption of infinite number of snapshots. In Fig. 7, we observe that the MRA, which has the largest aperture, achieves the lowest MSE. The co-prime array, which has the smallest aperture, has higher MSE than the MRA and two nested arrays. In Fig. 8, we observe that the MSE of the co-prime array is significantly higher than the other three arrays. This is because the co-prime array is the only array among the four arrays whose difference coarray is not a full ULA. Consequently, the central ULA part of the co-prime array is the smallest among the four, resulting a significantly higher MSE.



Fig. 3. $|MSE_{em} - MSE_{em}|/MSE_{em}$ for different types of arrays under different numbers of snapshots and different magnitudes of perturbations. The results are averaged from 3000 trials.



Fig. 4. RMSE vs. perturbation level for four different sparse linear arrays with the same number of sensors. The empirical results are averaged from 1000 trials.



Fig. 5. RMSE vs. perturbation level for four different sparse linear arrays with the same aperture. The empirical results are averaged from 1000 trials.

C. Numerical Results of the CRB

We close this section with numerical results of the CRB we derived in Section V. We demonstrate that the CRB obtained from Proposition 2 is indeed achievable in cases when the number of sources is greater than the number of sensors. We consider 11 sources evenly distributed between $-\pi/3$ and $\pi/3$ and fix the number of snapshots to 5000. We consider the first set of sparse linear arrays with the same number of sensors. We compare the CRB and the empirical MSE obtained by solving the following stochastic maximum likelihood problem



Fig. 6. RMSE vs. SNR for Co-prime (2,3) under different perturbation levels. The empirical results are averaged from 1000 trials.



Fig. 7. Empirical RMSEs vs. analytical approximations under different numbers of snapshots for four different sparse linear arrays with the same number of sensors, based on the stochastic error model. The empirical results are averaged from 5000 trials.

using the optimization toolbox in MATLAB:

$$\min_{\boldsymbol{\theta}, \boldsymbol{p}, \boldsymbol{\sigma}_{\mathbf{n}}^2, \boldsymbol{\delta}} \log \det(\tilde{\boldsymbol{R}}(\boldsymbol{\theta}, \boldsymbol{p}, \boldsymbol{\sigma}_{\mathbf{n}}^2, \boldsymbol{\delta})) + \operatorname{tr}(\tilde{\boldsymbol{R}}^{-1}(\boldsymbol{\theta}, \boldsymbol{p}, \boldsymbol{\sigma}_{\mathbf{n}}^2, \boldsymbol{\delta})\hat{\boldsymbol{R}}),$$

where $\tilde{R}(\theta, p, \sigma_{\rm n}^2, \delta)$ follows the definition in (8).

The results are plotted in Fig. 9. For comparison, we also include the CRB without considering sensor location errors [16]. We first notice that the CRB converges to a positive constant as SNR increases, which agrees with our analysis of the CRB in the underdetermined case in Section V. We



Fig. 8. Empirical RMSEs vs. analytical approximations under different numbers of snapshots for four different sparse linear arrays with the same aperture, based on the stochastic error model. The empirical results are averaged from 5000 trials.

then observe that, given sufficient SNR, the MSE of the MLE indeed achieves the CRB for all four arrays. Additionally, there is a significant gap between the values of the CRB when the sensor location errors are considered and when they are not. This gap shows that unknown sensor location errors have a drastic impact on the achievable DOA estimation performance of sparse linear arrays.

VII. CONCLUDING SUMMARY

We statistically analyzed the performance of SS-MUSIC in the presence of sensor location errors for general sparse linear arrays. We derived a closed-form expression of the asymptotic MSE of SS-MUSIC in the presence of small and deterministic sensor location errors, as well as the CRB for joint estimation of DOAs and sensor location errors. We also gave a brief extension of our results to the stochastic sensor location error model, and analyzed the Gaussian case. Our results will benefit future research on the development of robust DOA estimators using sparse linear arrays and the optimal design of sparse linear arrays. When investigating the case of the stochastic error model, we assumed that the time-variant sensor location errors are i.i.d. This assumption, while providing convenience in statistical analysis, may not hold in practice. In the future, we will extend our analysis for the stochastic error model by introducing motion models for the sensor location errors and utilizing the tools in semiparametric modeling. It would also be of great interest to further analyze the CRB for joint estimation of DOAs and sensor location errors.

APPENDIX A Derivation of the FIM

The (m, n)-th element of the single snapshot FIM for the observation model (6) is given by [2], [27]

$$\boldsymbol{J}_{mn} = \operatorname{tr}\left[\frac{\partial \tilde{\boldsymbol{R}}}{\partial \eta_m} \tilde{\boldsymbol{R}}^{-1} \frac{\partial \tilde{\boldsymbol{R}}}{\partial \eta_n} \tilde{\boldsymbol{R}}^{-1}\right]$$

Using the properties that $tr(AB) = vec(A^T)^T vec(B)$, and that $vec(AXB) = (B^T \otimes A) vec(X)$ [31], we can express the FIM as (33).

To obtain the FIM, we need to evaluate the partial derivatives in (34). The partial derivatives of \tilde{r} with respect to θ , p, and σ_n^2 have been derived in [16], [36], [37]. We will focus on deriving the partial derivatives of \tilde{r} with respect to the sensor location errors, making use of the following lemma:

Lemma 2. Let $A, B \in \mathbb{C}^{M \times K}$, $e \in \mathbb{C}^M$, and $p \in \mathbb{C}^K$. Then

$$(\boldsymbol{A} \odot \boldsymbol{e} \boldsymbol{e}^T \boldsymbol{B}) \boldsymbol{p} = (\boldsymbol{A} \boldsymbol{P} \boldsymbol{B}^T \boldsymbol{e}) \otimes \boldsymbol{e},$$

 $(\boldsymbol{e} \boldsymbol{e}^T \boldsymbol{B} \odot \boldsymbol{A}) \boldsymbol{p} = \boldsymbol{e} \otimes (\boldsymbol{A} \boldsymbol{P} \boldsymbol{B}^T \boldsymbol{e}),$

where $\boldsymbol{P} = \operatorname{diag}(\boldsymbol{p})$.

Proof. For brevity, we show only the proof of the first equality. The proof of the second equality follows the same idea. By the definition of the Khatri-Rao product and the fact that $\boldsymbol{a} \otimes \boldsymbol{b} = \text{vec}(\boldsymbol{b}\boldsymbol{a}^T)$, the left hand side can be expressed as

$$\sum_{i} p_i(\boldsymbol{a}_i \otimes \boldsymbol{e}\boldsymbol{e}^T \boldsymbol{b}_i) = \sum_{i} p_i \operatorname{vec}(\boldsymbol{e}\boldsymbol{e}^T \boldsymbol{b}_i \boldsymbol{a}_i^T). \quad (36)$$

Because the Kronecker product follows the distributive rule, the right hand side is given by

$$\left(\sum_{i} p_{i} \boldsymbol{a}_{i} \boldsymbol{b}_{i}^{T} \boldsymbol{e}\right) \otimes \boldsymbol{e} = \sum_{i} p_{i} (\boldsymbol{a}_{i} \boldsymbol{b}_{i}^{T} \boldsymbol{e} \otimes \boldsymbol{e})$$
$$= \sum_{i} p_{i} \operatorname{vec}(\boldsymbol{e} \boldsymbol{e}^{T} \boldsymbol{b}_{i} \boldsymbol{a}_{i}^{T}),$$
(37)

which is equal to the left hand side.

Because the partial derivative of Khatri-Rao products follows the Leibniz rule, we have

$$\frac{\partial \tilde{\boldsymbol{r}}}{\partial u_{i}} = \frac{\partial}{\partial u_{i}} [(\tilde{\boldsymbol{A}}^{*} \odot \tilde{\boldsymbol{A}})\boldsymbol{p} + \sigma_{n}^{2} \operatorname{vec}(\boldsymbol{I}_{M})] \\
= \left(\frac{\partial \tilde{\boldsymbol{A}}^{*}}{\partial u_{i}} \odot \tilde{\boldsymbol{A}} + \tilde{\boldsymbol{A}}^{*} \odot \frac{\partial \tilde{\boldsymbol{A}}}{\partial u_{i}}\right) \boldsymbol{p} \\
= \left\{ [\boldsymbol{e}_{M}^{(i)}(\boldsymbol{e}_{M}^{(i)})^{T} \tilde{\boldsymbol{A}}_{u}^{*}] \odot \tilde{\boldsymbol{A}} + \tilde{\boldsymbol{A}}^{*} \odot [\boldsymbol{e}_{M}^{(i)}(\boldsymbol{e}_{M}^{(i)})^{T} \tilde{\boldsymbol{A}}_{u}] \right\} \boldsymbol{p}$$
(38)

By Lemma 2, we immediately obtain that

$$\frac{\partial \tilde{\boldsymbol{r}}}{\partial u_i} = (\tilde{\boldsymbol{A}}^* \boldsymbol{P} \tilde{\boldsymbol{A}}_u^T \boldsymbol{e}_M^{(i)}) \otimes \boldsymbol{e}_M^{(i)} + \boldsymbol{e}_M^{(i)} \otimes (\tilde{\boldsymbol{A}} \boldsymbol{P} \tilde{\boldsymbol{A}}_u^H \boldsymbol{e}_M^{(i)}).$$
(39)

Combining (39) with the definition of the Khatri-Rao product leads to (35c). The derivation of (35d) follows the same idea.

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Fig. 9. CRB versus the empirical MSE of the maximum likelihood estimator for four different sparse linear arrays under different SNRs. The empirical MSEs are averaged from 500 trials.

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